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An Introduction to Localisation and Supersymmetry in Curved Space

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This paper presents the lecture notes of a course that I taught at the Ninth Modave Summer School in Mathematical Physics. The course has been designed to give an introduction to new exact results for supersymmetric field theories obtained by localisation of the path integral. I first review localisation theorems for ordinary integrals and the extension of these ideas to supersymmetric quantum field theories. Then I introduce the formalism of rigid supersymmetry in curved space and apply it to three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on the round 3-sphere. Finally I sketch the derivation of the localisation formula for the partition functions of such theories.

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1. Introduction

The full information on a quantum field theory is encoded in its Feynman path integral, an infinite-dimensional integral on the space of Euclidean field configurations with a weight $e^{-\frac{S}{\hbar}}$ determined by its Euclidean action. Ideally, the path integral is the key object that we would like to compute to study a quantum field theory. This is extremely hard to do in practice, because of the daunting range of integration. In our first course on quantum field theory, we all learn how to approximate the path integral by means of perturbative expansions about free theories. Unfortunately this method is only valid for weak coupling. Even if we are able to resum the perturbative series at weak coupling, typically this series is only asymptotic and has vanishing radius of convergence in the coupling. Only after taking into account all the non-perturbative corrections we obtained a well-defined result as a function of the coupling.

In view of these limitations, it is natural to look for quantum field theories where the path integral can be computed exactly. Apart from free field theories, where the integral is Gaussian, the only known examples until a few years ago were topological or cohomological field theories defined on compact manifolds, such as for instance Chern-Simons theories. This class includes topologically twisted supersymmetric field theories [1, 2], which know about holomorphic sectors of the ordinary supersymmetric field theories in curved space.

Over the last few years, a wealth of new exact results on path integrals has become available in the context of supersymmetric field theories placed on curved backgrounds, starting from $4d$ $\mathcal{N}=2$ gauge theories in the Omega background [3], and particularly after the work of Pestun [4], who defined and computed the path integral of $4d$ $\mathcal{N}=2$ gauge theories on the 4-sphere. This work was later extended to various types of rigid supersymmetric field theories on curved compact manifolds. Rigid supersymmetric field theories on compact manifolds considerably enlarge the class of computable quantum field theories from the old topologically twisted supersymmetric theories, and give access to non-holomorphic information such as correlators of conserved currents.

The key technique that allows the exact computation of the path integral, both for topologically twisted supersymmetric theories and for the more general rigid supersymmetric theories defined on curved spaces is *supersymmetric localisation*, which gives rise to very powerful fixed point theorems. Localisation relies on supersymmetry to prove that the path integral only receives contribution from the locus of fixed points of supersymmetry (the localisation locus). As we will see, supersymmetric localisation is a natural generalisation of equivariant localisation in the context of ordinary integrals which have bosonic symmetries with fixed points. In that simpler framework one shows that certain finite-dimensional integrals only receive contributions from the fixed points of the action of the symmetry group.

The power of localisation is to reduce the dimensionality of the integrals that we need to compute. In the case of the path integral of a quantum field theory in D dimensions, localisation formulae can reduce the path integral on D-dimensional fields to a path integral on lower d-dimensional fields. If in particular the localisation locus consists of constant field configurations only ($d=0$), we are left with a finite-dimensional integral (the path integral of a zero-dimensional field theory), which it is often possible to evaluate exactly. In even more favourable cases one can further reduce the dimensionality of these ordinary integrals, and sometimes collapse them to a discrete sum over fixed points in field space.

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Stated in physical terms, localisation formulae can be viewed as instances where the semi-classical 1-loop/WKB/stationary phase approximation is exact. The crucial point is that this exact saddle point approximation is not for the original action with quantum parameter $\hbar$, but rather for a modified action with a deformation term weighted by an auxiliary quantum parameter $\hbar_{aux} = 1/t$, which does not change the result of the path integral and can therefore be sent to zero.

Localisation formulae provide non-perturbative exact results that can be used to extract physical or mathematical information on the quantum field theory, to compute expectation values and correlators of certain observables, and to test or infer non-perturbative dualities.

These lecture notes are organised as follows. In section 2, after recalling the stationary phase approximation, I introduce equivariant cohomology and equivariant localisation formulae for finite-dimensional integrals, focussing on the case of Abelian symmetries for simplicity. In section 3 I introduce the localisation argument for supersymmetric quantum field theories, drawing a parallel with the localisation argument in equivariant cohomology. In section 4 I explain how to define supersymmetric theories in curved space, and apply the formalism to three-dimensional $\mathscr{N}=2$ supersymmetric gauge theories on the round 3-sphere. Field theories on compact manifolds are free of infrared divergences, therefore their path integrals are well defined and we can apply to them the technique of localisation. In the final section 5 I sketch an example of localisation of a supersymmetric quantum field theory on a compact manifold, again focussing on three-dimensional $\mathscr{N}=2$ supersymmetric gauge theories on the round 3-sphere to make contact with Antonio Amariti's lectures in the same school.

These lectures only cover the basic concepts of localisation and supersymmetry in curved space. We will only scratch the surface of the vast literature on both subjects. I have learnt this material from various sources, to which I will refer in the appropriate sections. My aim here is not to be original, but rather to introduce the necessary background to follow this rapidly developing line of research.

2. Basics of equivariant localization

The aim of this section is to introduce equivariant localisation theorems for finite-dimensional integrals. I start by recalling the stationary phase approximation and presenting an example where the stationary phase approximation is exact in section 2.1. In section 2.2 I introduce the notion of Abelian equivariant cohomology and equivariant integrals. In section 2.3 I show that equivariant integrals localise to the locus of fixed points of the equivariant action. In section 2.4 I sketch the derivation of the equivariant localisation theorem in the case where the fixed points are isolated, which generalises and implies the celebrated Duistermaat-Heckman exact stationary phase formula. Finally in section 2.5 I present the localisation formula for non-isolated fixed points.

Several informative reviews are available on this subject. I have largely drawn from [5, 6, 7], to which I refer the readers eager for more background.

2.1 Stationary phase approximation

Let us consider a smooth (compact) 2ℓ-dimensional Riemannian manifold $(\mathscr{M}, g)$ with local coordinates x, and a real smooth function $f(x)$. We wish to compute the oscillatory integral

$$Z_f(t) = \int_{\mathscr{M}} d^{2\ell}x \sqrt{\det g(x)}\; e^{itf(x)} \ . \tag{2.1}$$

In the $t \to \infty$ limit, the phase $tf(x)$ oscillates very rapidly and leads to destructive interference unless the phase is stationary. Therefore the leading contributions to the integral arise from the stationary points of f. Let us assume that f is a Morse function, namely that the stationary points of f $\{x_k|\, df(x_k)=0\}$ are isolated. Taylor expanding the phase $tf(x)$ about each stationary point x_k, the leading contribution to the oscillatory integral (2.1) is given by a sum of Gaussian integrals of quadratic fluctuations about each stationary point,

$$Z_f(t) = \left(\frac{2\pi i}{t}\right)^{\ell} \sum_{x_k:\, df(x_k)=0} (-i)^{\lambda(x_k)} \frac{e^{itf(x_k)}}{\sqrt{\det(g^{-1}H_f(x_k))}} + \mathscr{O}(t^{-\ell-1}) \ , \tag{2.2}$$

where $H_f(x)$ is the Hessian matrix of f at x and $\lambda(x_k)$ is the number of negative eigenvalues of $g^{-1}H_f(x_k)$, also known as the Morse index of f at x_k.

The stationary phase approximation (2.2) is familiar from quantum physics. If we view $1/t \sim \hbar$ as a quantum expansion parameter and $f \sim S$ as an action, (2.2) is a semiclassical approximation where each classical contribution $e^{\frac{i}{\hbar}S(x_k)}$ is weighted by a 1-loop determinant of quadratic fluctuations. Higher loop corrections are down by positive powers of $1/t \sim \hbar$ and are subleading in the semiclassical limit $1/t \sim \hbar \to 0$.

It is instructive to consider an illustrative example, where $\mathscr{M} = S^2$ is the round 2-sphere of unit radius, with metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{2.3}$$

in spherical coordinates, and f is the *height function*

$$f(\theta,\varphi) = \cos\theta \tag{2.4}$$

measuring the value of the z coordinate for a 2-sphere centred at the origin of $\mathbb{R}^3$, see fig. 1. The

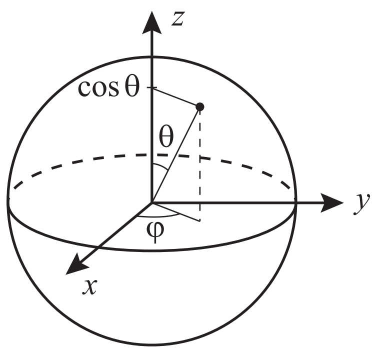

Figure 1: The round 2-sphere embedded in $\mathbb{R}^3$.

oscillatory integral (2.1) is easily evaluated:

$$\begin{aligned} Z_f(t) &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\, e^{it\cos\theta} = 2\pi \int_{-1}^{1} d(\cos\theta)\, e^{it\cos\theta} = \\ &= \frac{2\pi i}{t}(-e^{it} + e^{-it}) = 4\pi \frac{\sin t}{t} \ . \end{aligned} \tag{2.5}$$

A couple of remarks are in order:

1. The first entry in the second line of (2.5) expresses the result as a sum of two terms, corresponding to the two stationary points of the height function (2.4), the north pole and the south pole. The stationary phase approximation (2.2) captures the exact result, valid for any value of t.

2. We have used the $U(1)$ symmetry corresponding to azimuthal rotations $\varphi \to \varphi + c$ to separate φ and θ. The north and south poles, the stationary points of f, are fixed points of this circle action, see fig. 2. One says that the integral localises at the fixed points of the circle action. The result only depends on data of the circle action in an infinitesimal neighbourhood of the fixed points.[1]

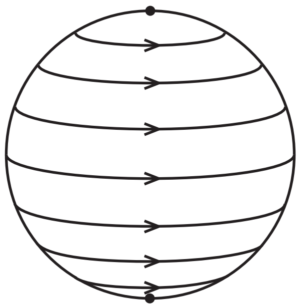

Figure 2: The azimuthal rotation which fixes the poles of the 2-sphere.

The idea of localisation originated with the seminal work of Duistermaat and Heckman [8], who discovered a class of phase space integrals with circle actions where the stationary phase approximation is exact and leads to a localisation formula which we will review in section 2.4.1. We will see that the crucial property for Duistermaat-Heckman formulae is the presence of a symmetry group with fixed points: the integral is given by a sum over contributions from the fixed points of the group action. As such, the Duistermaat-Heckman localisation formula can be viewed as a particular example of a more general equivariant localisation formula [9, 10, 11, 12], that we will discuss in section 2.4.

2.2 Abelian equivariant cohomology

We are interested in computing integrals over a manifold $\mathscr{M}$ with a symmetry group G, the first guess would be to reduce the integrals over $\mathscr{M}$ to integrals over the orbit space $\mathscr{M}/G$, as in the first line of (2.5). However, if G does not act freely but has fixed points, this quotient is not a smooth manifold, but rather an orbifold with singularities corresponding to the fixed points, therefore we

[1]The round S^2 has $SO(3)$ isometry, but this larger symmetry is not needed to obtain the result. If we consider any $U(1)$-invariant monotonic function $f(\theta)$ with the same asymptotics of $\cos\theta$ at $\theta = 0, \pi$, the oscillatory integral (2.1) leads to the same result as (2.5).

cannot use the ordinary notions of differential forms and cohomology. The notion of G-equivariant cohomology of $\mathscr{M}$ generalises cohomology of $\mathscr{M}/G$ in a well defined manner, which makes sense even when G does not act freely. In the following for simplicity we will take G to be abelian.

We will work for simplicity with a 2ℓ-dimensional Riemannian manifold without boundary $(\mathscr{M}, g)$. Let $V = V^\mu \partial_\mu$ be a Killing vector field,

$$\mathscr{L}_V g_{\mu\nu} = 0 \quad \Longleftrightarrow \quad \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0\ , \tag{2.6}$$

with $\mathscr{L}_V$ the Lie derivative along the integral curves of V, and assume that it generates a $U(1)$ isometry, namely that it has closed orbits with a common period. I will loosely identify the $U(1)$ group with its associated Killing vector V in the following.

Let us consider the space $\bigwedge \mathscr{M} = \{\alpha = \sum_{n=0}^{2\ell} \alpha_n |\ \alpha_n \in \bigwedge^n \mathscr{M}\}$ of polyforms on $\mathscr{M}$, formal sums of differential forms of different degrees n, and define the *V-equivariant differential* $d_V : \bigwedge \mathscr{M} \to \bigwedge \mathscr{M}$ as

$$d_V := d - \imath_V\ . \tag{2.7}$$

Here d is the ordinary exterior differential $d:\ \bigwedge^n \mathscr{M} \to \bigwedge^{n+1} \mathscr{M}$, which increases the differential degree by 1, and $\imath_V:\ \bigwedge^n \mathscr{M} \to \bigwedge^{n-1} \mathscr{M}$ is the contraction with the vector field V, which decreases the differential degree by 1. It is possible to introduce a parameter ξ of formal degree 2 in front of $\imath_V$ in (2.7) to make d_V homogeneous of degree 1, but in the abelian case ξ will not play an important rôle, therefore we set it to 1 in our discussion.

On the space of polyforms $\bigwedge \mathscr{M}$, the equivariant differential squares to the Lie derivative,

$$d_V^2 = d^2 - d\imath_V - \imath_V d + \imath_V^2 = -(d\imath_V + \imath_V d) = -\mathscr{L}_V\ , \tag{2.8}$$

because $d^2 = 0$ and $\imath_V \imath_V \alpha_n = 0$ for a form α_n of any degree n since forms are antisymmetric. In particular on the space of *V-equivariant polyforms* $\bigwedge_V \mathscr{M} = \{\alpha \in \bigwedge \mathscr{M} | \mathscr{L}_V \alpha = 0\}$ the equivariant differential d_V behaves as a coboundary operator: $d_V^2 = 0$. We can therefore restrict the action of the equivariant differential d_V to $\bigwedge_V \mathscr{M}$ and use it to define a *V-equivariant de Rham cohomology*. A polyform α is called *equivariantly closed* if $d_V \alpha = 0$. A polyform β is called *equivariantly exact* if $\beta = d_V \gamma$ for a well-defined polyform γ. The *n-th V-equivariant de Rham cohomology group of $\mathscr{M}$* is then defined as

$$H_V^n(\mathscr{M}) = \frac{\ker d_V|_{\bigwedge_V^n \mathscr{M}}}{\mathrm{Im}\ d_V|_{\bigwedge_V^{n-1} \mathscr{M}}} \tag{2.9}$$

As mentioned above, V-equivariant cohomology groups of $\mathscr{M}$ agree with ordinary cohomology groups of $\mathscr{M}/U(1)_V$ when the circle action $U(1)_V$ generated by V acts freely, and provide a well-defined extension to cases where the $U(1)_V$ action on $\mathscr{M}$ has fixed points.

By the definition (2.7), $(d_V \alpha)_n = d\alpha_{n-1} - \imath_V \alpha_{n+1}$, which involves forms of degree differing by 2 on the right-hand side. This leads to recursive relations for terms of different degrees of equivariantly closed forms. Even and odd degree parts do not talk to each other and appear in different systems of recursive relations. Moreover, if a polyform α of highest differential degree n is equivariantly closed, then α_n and α_{n-1} are closed forms in the ordinary sense. Similarly, if a polyform β of highest differential degree n is equivariantly exact, then β_n and β_{n-1} are exact in the ordinary sense.

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Next, we define *equivariant integrals of polyforms* over $\mathscr{M}$ as integrals of their top form terms:

$$\int_{\mathscr{M}} \alpha := \int_{\mathscr{M}} \alpha_{2\ell} . \tag{2.10}$$

Because the top term of an equivariantly exact polyform is exact, it follows immediately by the definition (2.10) and Stokes' theorem that integrals of equivariantly exact polyforms vanish:

$$\int_{\mathscr{M}} d_V \beta = \int_{\mathscr{M}} d\beta_{2\ell-1} = 0 , \tag{2.11}$$

therefore integrals only depend on the equivariant cohomology class of the integrand:

$$\int_{\mathscr{M}} (\alpha + d_V \beta) = \int_{\mathscr{M}} \alpha . \tag{2.12}$$

2.3 Equivariant integrals localise

Our aim for the rest of this section will be to evaluate integrals of equivariantly closed polyforms over $\mathscr{M}$: $\int_{\mathscr{M}} \alpha$ with $[\alpha] \in H_V^*(\mathscr{M})$. We will achieve this by means of equivariant localisation theorems [10, 11, 12]. The essence of equivariant localisation theorems is that integrals of V-equivariantly closed polyforms only receive contributions from an infinitesimal neighbourhood of the fixed point locus of $U(1)_V$, namely the zero locus of V:

$$\mathscr{M}_V = \{x \in \mathscr{M} \mid V|_x = 0\} . \tag{2.13}$$

It is customary to say that the integral on $\mathscr{M}$ *localises to* the fixed point locus $\mathscr{M}_V$. We can understand why these integrals localise in two ways, that I now explain.

1st localisation argument The first way involves showing a version of Poincaré's lemma: a V-equivariantly closed polyform on $\mathscr{M}$ is equivariantly exact on the complement of the fixed point locus, $\mathscr{M} \setminus \mathscr{M}_V$. The proof involves the 1-form η dual to the Killing vector V

$$\eta := g(V, \cdot) = V^\mu g_{\mu\nu} dx^\nu = V_\mu dx^\mu , \tag{2.14}$$

which is V-equivariant because V is a Killing vector. Its V-equivariant differential

$$d_V \eta = -|V|^2 + d\eta \tag{2.15}$$

has a 0-form term which is minus the norm squared of the vector field V, and a 2-form term. (2.15) is invertible on $\mathscr{M} \setminus \mathscr{M}_V$, where its inverse[2]

$$(d_V \eta)^{-1} = -\frac{1}{|V|^2} \left(1 - \frac{d\eta}{|V|^2}\right)^{-1} = -\frac{1}{|V|^2} \sum_{n=0}^{\ell} \left(\frac{d\eta}{|V|^2}\right)^n \tag{2.16}$$

is equivariantly closed and well-defined. We can then define the polyform

$$\Theta_V := \eta (d_V \eta)^{-1} \tag{2.17}$$

[2]The inverse of a polyform is computed using the geometric series formula $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$, which truncates at degree equal to the dimension of the manifold.

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which by construction is equivariant, satisfies $d_V\Theta_V = 1$ and is well-defined on $\mathscr{M}\setminus\mathscr{M}_V$. Now we are in business: on the complement of the fixed point locus $\mathscr{M}_V$ of the Killing vector field, any equivariantly closed polyform α can be written as

$$\alpha = 1\cdot\alpha = (d_V\Theta_V)\alpha = d_V(\Theta_V\alpha)\ . \tag{2.18}$$

This shows that V-equivariantly closed forms on $\mathscr{M}$ are equivariantly exact on $\mathscr{M}\setminus\mathscr{M}_V$. By Stokes' theorem (2.11), the integral $\int_{\mathscr{M}}\alpha$ does not receive contributions from $\mathscr{M}\setminus\mathscr{M}_V$.

Exercise 2.1. Consider the oscillatory integral (2.5) on the round 2-sphere. The Killing vector that generates azimuthal rotations is $V=\partial_\varphi$. Compute: 1) the dual 1-form $\eta=\sin^2\theta\, d\varphi$ and the norm squared $|V|^2=\sin^2\theta$ using the metric (2.3); 2) the polyform (2.17), $\Theta_V=-d\varphi$. Finally, check that the integrand of (2.5) can be traded for the equivariantly closed polyform

$$\alpha = e^{it\cos\theta}\left(\frac{1}{it}+d\varphi\wedge d\cos\theta\right) = \frac{1}{it}\, e^{it(\cos\theta+d\varphi\wedge d\cos\theta)}\ . \tag{2.19}$$

2nd localisation argument The second method to understand that the integral (2.10) localises to $\mathscr{M}_V$ is more direct. We have seen that the integral depends only on the equivariant cohomology class $[\alpha]$. Therefore we are free to use another deformed representative

$$\alpha_t := \alpha\, e^{td_V\beta}\ , \tag{2.20}$$

where t is a real parameter and β a V-equivariant polyform: $\mathscr{L}_V\beta=0$. For any such choice of β we have a 1-parameter family of representatives of the equivariant cohomology class $[\alpha]$. It should be clear that α_t is equivariantly cohomologous to $\alpha=\alpha_0$ because it is an equivariantly closed continuous deformation $\alpha=\alpha_0$. Explicitly, $\frac{d}{dt}\alpha_t=\alpha(d_V\beta)e^{td_V\beta}=d_V\left(\alpha\beta e^{td_V\beta}\right)$ is equivariantly exact, and similarly one shows that the finite difference $\alpha_t-\alpha_0$ is equivariantly exact. Therefore by Stokes' theorem (2.11)

$$\int_{\mathscr{M}}\alpha = \int_{\mathscr{M}}\alpha_t = \int_{\mathscr{M}}\alpha\, e^{td_V\beta} \qquad \forall\, t\ . \tag{2.21}$$

The integral on the right-hand side of (2.21) takes its original form for $t=0$, but because it is independent of t, we are free to evaluate it for any value of t that makes the computation easier. In particular we can consider the limit $t\to+\infty$. If this limit exists, which is the case if the 0-form term of the exponent $(d_V\beta)_0$ is negative semi-definite with maximum equal to 0, it will compute the original integral $\int_{\mathscr{M}}\alpha$. In this limit the integral is dominated by the minima (which are also zeros) of $-(d_V\beta)_0$. Let us choose for instance $\beta=\eta$, the 1-form (2.14) dual to the Killing vector V. Plugging its equivariant differential (2.15) into the integral (2.21) and taking the limit $t\to+\infty$, we find

$$\int_{\mathscr{M}}\alpha = \lim_{t\to+\infty}\int_{\mathscr{M}}\alpha\, e^{td\eta}\, e^{-t|V|^2}\ . \tag{2.22}$$

The factor $e^{td\eta}$ is the exponential of a 2-form. Writing it out as a Taylor series, we see that it is in fact a polynomial of degree ℓ in t, because forms of degree higher than the dimension 2ℓ of the manifold vanish. Conversely, the factor $e^{-t|V|^2}$ is a genuine exponential: it provides a Gaussian factor that is increasingly peaked at $\mathscr{M}_V$ as $t\to+\infty$, and becomes in this limit a delta-function

$\delta(V)$ supported on the zero locus $\mathscr{M}_V$ of the Killing vector V. The limit $t \to +\infty$ therefore shows that the integral localises to the locus $\mathscr{M}_V$.

Note that there is some freedom in the previous localisation arguments:

1. If the manifold has several Killing vectors $V_1, V_2, \ldots, V_r$ under which α is equivariantly closed, we can localise to $\mathscr{M}_{V_1}$, $\mathscr{M}_{V_2}, \ldots$ or $\mathscr{M}_{V_r}$, or even to their intersection.

2. We can run the second localisation argument for any equivariant polyform β such that $-(d_V\beta)_0$ is positive semi-definite with absolute minimum 0. Taking $t \to \infty$, the integral (2.21) localises to the zeros of $-(d_V\beta)_0$. The freedom in choosing β is often referred to as the choice of a *localisation scheme*. Different localisation schemes yield different localisation loci and can provide different localisation formulae for the same integral.

2.4 Localisation formula for equivariant integrals: isolated fixed points

As we have seen, understanding that integrals (2.10) of equivariantly closed polyforms localise is relatively easy. Computing the contribution to the integral from the localisation locus is a bit more involved. For the sake of simplicity, we will evaluate (2.22) following [9] (see also [7]) under the assumption that the localisation locus, the zero locus of the Killing vector V, is a set of isolated fixed points: $\mathscr{M}_V = \{x_k\}$. Let us zoom near a zero P of the vector field V. Locally, we can work with an adapted 'inertial' Cartesian coordinate system $(x_i = r_i \cos\varphi_i, y_i = r_i \sin\varphi_i)_{i=1}^{\ell}$ with origin at P so that the metric reads

$$ds^2 \simeq \sum_{i=1}^{\ell}(dx_i^2 + dy_i^2) = \sum_{i=1}^{\ell}(dr_i^2 + r_i^2 d\varphi_i^2) \tag{2.23}$$

and the Killing vector reads

$$V \simeq \sum_{i=1}^{\ell} \omega_{P,i}\left(-y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i}\right) = \sum_{i=1}^{\ell} \omega_{P,i}\frac{\partial}{\partial \varphi_i} \,. \tag{2.24}$$

The circle action generated by the Killing vector V on the i-th eigenspace in the tangent space $T_P\mathscr{M}$ is a rotation $R_i(\phi_i)$ with weight (or "angular velocity") $\omega_{P,i}$

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \to R_i(\phi_i)\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos(\omega_{P,i}\phi_i) & \sin(\omega_{P,i}\phi_i) \\ -\sin(\omega_{P,i}\phi_i) & \cos(\omega_{P,i}\phi_i) \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} , \tag{2.25}$$

see fig. 3. Its infinitesimal action is given by

$$L_V = R_i^{-1}(\phi_i)\frac{dR_i(\phi_i)}{d\phi_i} = \begin{pmatrix} 0 & \omega_{P,i} \\ -\omega_{P,i} & 0 \end{pmatrix} . \tag{2.26}$$

The dual 1-form η is locally

$$\eta \simeq \sum_{i=1}^{\ell} \omega_{P,i}(-y_i dx_i + x_i dy_i) = \sum_{i=1}^{\ell} \omega_{P,i} r_i^2 d\varphi_i \tag{2.27}$$

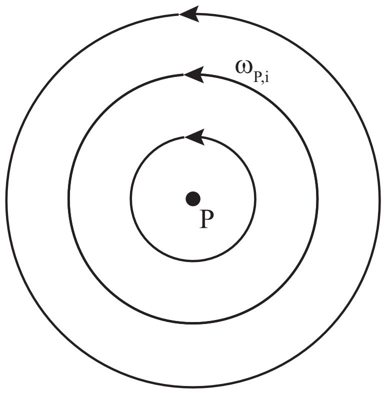

Figure 3: Circle action of the Killing vector V on the i-th eigenspace in the tangent space $T_P\mathscr{M}$ to a fixed point P.

and its equivariant differential is

$$
\begin{aligned}
d_V\eta &\simeq 2\sum_{i=1}^{\ell}\omega_{P,i}dx_i\wedge dy_i-\sum_{i=1}^{\ell}\omega_{P,i}^2(x_i^2+y_i^2)= \\
&=\sum_{i=1}^{\ell}\omega_{P,i}d(r_i^2)\wedge d\varphi_i-\sum_{i=1}^{\ell}\omega_{P,i}^2r_i^2\ .
\end{aligned}
\tag{2.28}
$$

So the local contribution to $\int_{\mathscr{M}}\alpha$ from an infinitesimal neighbourhood N_P of the fixed point P of the $U(1)$ action can be written as

$$
\begin{aligned}
\lim_{t\to+\infty}\int_{N_P}\alpha\ e^{td_V\eta} &= \lim_{t\to+\infty}\alpha_0(P)\prod_{i=1}^{\ell}\left(2t\ \omega_{P,i}\int_{\mathbb{R}^2}dx_i\wedge dy_i\ e^{-t\omega_{P,i}^2(x_i^2+y_i^2)}\right)= \\
&=\lim_{t\to+\infty}\alpha_0(P)\prod_{i=1}^{\ell}\left(t\ \omega_{P,i}\int_0^{2\pi}d\varphi_i\int_0^{\infty}d(r^2)e^{-t\omega_{P,i}^2r_i^2}\right)= \\
&=\alpha_0(P)\frac{(2\pi)^\ell}{\prod_{i=1}^{\ell}\omega_{P,i}}\ .
\end{aligned}
\tag{2.29}
$$

In the first line of (2.29) we have only kept the leading term as $t\to+\infty$, which comes from the degree 2ℓ form in the Taylor expansion of $\exp(2t\ \omega_{P,i}dx_i\wedge dy_i)$, that multiplies the scalar term α_0 in α. Moreover in the limit the Gaussian integrand is supported on a smaller and smaller neighbourhood of P of radius proportional to $t^{-1/2}$, which tends to a delta-function. Therefore the integral on the neighbourhood N_P can be traded for an integral on affine flat space $\mathbb{R}^{2\ell}$. The result of the Gaussian integral is directly proportional to $\alpha_0(P)$, the value of the scalar term of the polyform α evaluated at the fixed point, and inversely proportional to the product of "angular velocities" $\prod_{i=1}^{\ell}\omega_{P,i}$. Having in mind the circle action (2.25) on the Cartesian coordinates (x_i,y_i), $\prod_{i=1}^{\ell}\omega_{P,i}$ is the Pfaffian of (minus) the infinitesimal circle action L_V (2.26) of V on $T_P\mathscr{M}$.[3]

Adding up the contributions to $\int_{\mathscr{M}}\alpha$ of all the fixed points $x_k\in\mathscr{M}_V$, we find the Atiyah-Bott-

[3]The Pfaffian of a 2ℓ-dimensional antisymmetric matrix M is $\mathrm{Pf}\,M=\varepsilon^{i_1i_2\dots i_{2\ell-1}i_{2\ell}}M_{i_1i_2}\dots M_{i_{2\ell-1}i_{2\ell}}$.

Berline-Vergne localisation formula [11, 12]

$$\int_{\mathscr{M}} \alpha = (2\pi)^{\ell} \sum_{x_k \in \mathscr{M}_V} \frac{\alpha_0(x_k)}{\prod_{i=1}^{\ell} \omega_{x_k,i}} = (2\pi)^{\ell} \sum_{x_k \in \mathscr{M}_V} \frac{\alpha_0(x_k)}{\mathrm{Pf}\,(-L_V(x_k))} \tag{2.30}$$

for the integral of a polyform which is equivariantly closed under a circle action generated by a vector field V with isolated zeros.

2.4.1 Duistermaat-Heckman formula

The Duistermaat-Heckman localisation formula [8] for symplectic manifolds with a Hamiltonian circle action can be obtained as a corollary of the localisation formula (2.30).

Let us consider a 2ℓ-dimensional symplectic manifold $(\mathscr{M},\omega)$, that is an even-dimensional manifold $\mathscr{M}$ equipped with a nondegenerate closed 2-form ω, called symplectic form. An example of symplectic manifold that we are all familiar with is the classical phase space of an unconstrained particle in ℓ flat dimensions. The phase space is $\mathbb{R}^{2\ell}$, with coordinates $(q^i,p_i)_{i=1}^{\ell}$, where q^i are coordinates in the configuration space $\mathbb{R}^{\ell}$ and p_i are the conjugate momenta. The symplectic form is $\omega_0 = \sum_{i=1}^{\ell} dp_i \wedge dq^i$. Slightly less trivial examples of symplectic manifolds are total spaces of cotangent bundles T^*X, where q^i are coordinates on the base manifold X and p_i are coordinates on the cotangent fibres. Much like any n-dimensional differentiable manifold is locally diffeomorphic to $\mathbb{R}^n$, Darboux theorem guarantees that any 2ℓ-dimensional symplectic manifold $(\mathscr{M},\omega)$ is locally symplectomorphic to $(\mathbb{R}^{2\ell},\omega_0)$, where a symplectomorphism is a map between two symplectic manifolds which preserves the symplectic form (for physicists, a canonical transformation).

Let us assume that our symplectic manifold $(\mathscr{M},\omega)$ has a Hamiltonian H for a circle action, with V the associated Hamiltonian vector field. This means that the vector field V which generates the circle action is related to the Hamiltonian H by

$$dH = \iota_V \omega \quad \Longleftrightarrow \quad \partial_\nu H = V^\mu \omega_{\mu\nu} \ . \tag{2.31}$$

The Hamiltonian H is conserved along the flow of V: $\mathscr{L}_V H = V(H) = V^\mu \partial_\mu H = 0$, by the antisymmetry of the symplectic form. This Hamiltonian flow is analogous to the flow which describes time evolution in phase space in Hamilton's formulation of classical mechanics, except that no *time* evolution is implied here.

Now note that the definition (2.31) can be written as

$$d_V(H+\omega) = 0 \ , \tag{2.32}$$

so given a Hamiltonian vector field V with Hamiltonian H in a symplectic manifold $(\mathscr{M},\omega)$, we can construct a V-equivariantly closed polyform $H+\omega$. This observation and the localisation formula (2.30) allow us to evaluate the oscillatory integrals[4]

$$Z_H(t) = \int_{\mathscr{M}} \frac{\omega^\ell}{\ell!}\, e^{itH} \tag{2.33}$$

[4]Replacing $t \to i\beta$, we can evaluate canonical partition functions of classical Hamiltonian systems. For instance (2.5) leads to the partition function of a classical spin.

with Liouville measure $\frac{\omega^\ell}{\ell!}$. Indeed we can write

$$Z_H(t) = \frac{1}{(it)^\ell}\int_{\mathscr{M}} e^{it(H+\omega)} = \left(\frac{2\pi i}{t}\right)^\ell \sum_{x_k:\ dH(x_k)=0} \frac{e^{itH(x_k)}}{\mathrm{Pf}\, L_V(x_k)}\ , \tag{2.34}$$

where in the last equality we have used the Atiyah-Bott-Berline-Vergne formula (2.30) for $\alpha = e^{it(H+\omega)}$. This is the celebrated Duistermaat-Heckman formula, which shows that the stationary phase approximation is exact for the oscillatory integrals (2.33). Historically this formula was derived before the more general (2.30).

As an example, we can apply the Duistermaat-Heckman formula to the oscillatory integral on S^2 that we computed in (2.5). The symplectic form is the volume form $\omega = \sin\theta\ d\theta\wedge d\varphi = d\varphi\wedge d\cos\theta$, the Hamiltonian vector field is $V = \frac{\partial}{\partial\varphi}$ and its Hamiltonian is the height function $H = \cos\theta$. The Pfaffians of the infinitesimal action of V at the north pole and at the south pole are respectively $\mathrm{Pf}(-L_V(N)) = +1$ and $\mathrm{Pf}(-L_V(S)) = -1$, see fig. 4. Taking all the factors into

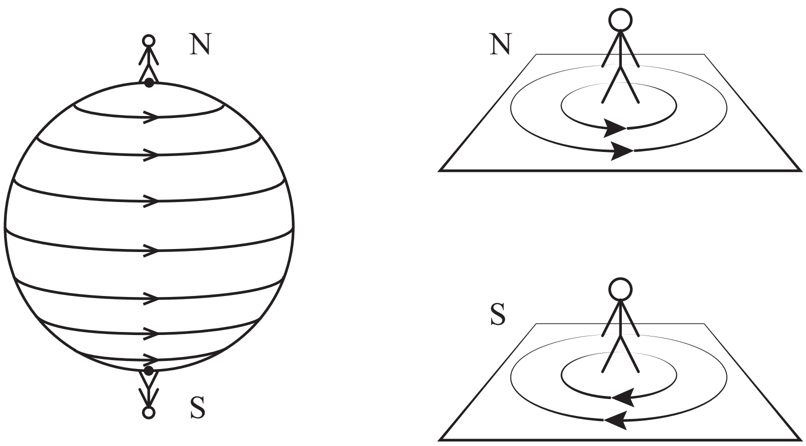

Figure 4: The Killing vector V flows counterclockwise around the north pole N and clockwise around the south pole S, with angular velocities $+1$ and -1.

account, the Duistermaat-Heckman formula (2.34) reproduces the first expression in the second line of (2.5), where the first term is the contribution from the north pole and the second term is the contribution from the south pole.

Exercise 2.2. Evaluate (2.34) for the non-compact symplectic manifold $\mathbb{R}^2$ with $\omega = dx\wedge dy = \frac{1}{2}d(r^2)\wedge d\varphi$ and Hamiltonian vector field $V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial\varphi}$ with Hamiltonian $H = -\frac{x^2+y^2}{2} = -\frac{r^2}{2}$. What do we need to assume about t in order to get the result? These integrals are also called equivariant symplectic volumes, so this is the equivariant symplectic volume of $\mathbb{R}^2$.

2.5 Localisation formula for equivariant integrals: non-isolated fixed points

In the previous subsection I assumed that the fixed points of the Killing or Hamiltonian vector field V are isolated. In this section, without entering into the details of the derivation, I will sketch the logic of the localisation formulae when the localisation locus $\mathscr{M}_V$ has continuous components as depicted in fig. 5.

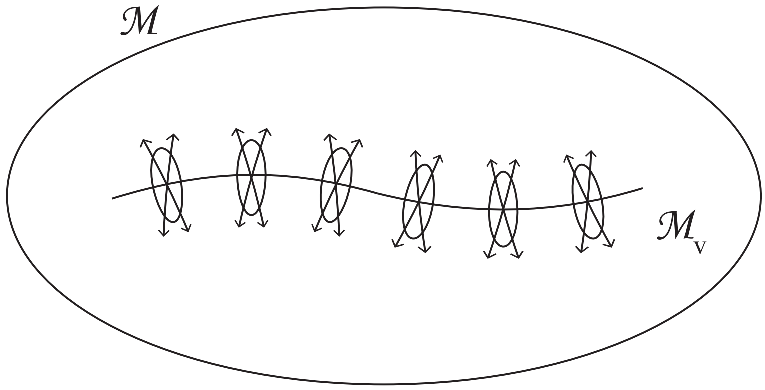

Figure 5: A manifold $\mathscr{M}$ with a non-isolated locus $\mathscr{M}_V$ of fixed points of a circle action. Fluctuation modes normal to $\mathscr{M}_V$ are depicted as outgoing arrows.

The idea is simple. We want to evaluate the RHS of (2.22), keeping into account all the components of $\mathscr{M}_V$. The following discussion applies component by component. Since we take the limit where the localisation parameter $t \to +\infty$ in the RHS of (2.22), the integral only receives contribution from an infinitesimal neighbourhood of $\mathscr{M}_V$, that can be chosen to be V-invariant. We then separate modes *normal* to the localisation locus $\mathscr{M}_V$ and modes *tangent* to the localisation locus. Recall that the tangent bundle to $\mathscr{M}_V$ splits into the tangent bundle of $\mathscr{M}_V$ and the normal bundle to $\mathscr{M}_V$: $T\mathscr{M}|_{\mathscr{M}_V} = T\mathscr{M}_V \oplus \mathscr{N}\mathscr{M}_V$. We are only interested in an infinitesimal neighbourhood of $\mathscr{M}_V$, so we can treat the normal modes infinitesimally and work with the normal bundle $\mathscr{N}\mathscr{M}_V$. The tangent modes should instead be treated non-linearly, so we work with $\mathscr{M}_V$.

The $t|V|^2$ term in the exponent of (2.22) is a potential for these fluctuations, which starts at quadratic order because it is minimised at $\mathscr{M}_V$. While the fluctuations tangent to $\mathscr{M}_V$ keep $|V|^2 = 0$ and are massless, the fluctuations normal to $\mathscr{M}_V$ appear in quadratic terms and are massive. We integrate the latter out in a semiclassical saddle point expansion with respect to the auxiliary quantum parameter $\hbar_{aux} = 1/t$. Scaling the normal fluctuations like $t^{-1/2}$, we see that the powers of t cancel out in the "1-loop" contribution and are negative for higher loops. Therefore the leading saddle point approximation which only includes the "1-loop" contribution is exact in the $t \to +\infty$ limit. We are then left with an integral on $\mathscr{M}_V$ which only involves tangent modes, subject to an "effective action" due to the original integrand (the "classical term") and the result of integrating out the massive normal modes ("1-loop term" in the auxiliary quantum expansion in $\hbar_{aux} = 1/t$).

The result of this computation is the Berline-Vergne localisation formula [10, 11]:

$$\int_{\mathscr{M}} \alpha = \int_{\mathscr{M}_V} \frac{i^*\alpha}{\chi_V^{\mathscr{N}\mathscr{M}_V}} = \int_{\mathscr{M}_V} \frac{i^*\alpha}{\mathrm{Pf}\left(-\frac{\Omega_\nabla + L_V^{\mathscr{N}\mathscr{M}_V}}{2\pi}\right)} . \tag{2.35}$$

Here $i : \mathscr{M}_V \hookrightarrow \mathscr{M}$ is the embedding of the localisation locus $\mathscr{M}_V$ in $\mathscr{M}$, and $i^*\alpha$ is the pull-back of α to the localisation locus, the "classical term" in the previous discussion. $\chi_V^{\mathscr{N}\mathscr{M}_V} = \mathrm{Pf}\left(-\Omega_V^{\mathscr{N}\mathscr{M}_V}/(2\pi)\right)$ is the V-equivariant Euler class of the normal bundle to $\mathscr{M}_V$, which is the Pfaffian of the V-equivariant curvature of the normal bundle to $\mathscr{M}_V$: this is the "1-loop determinant" of normal fluctuations in the previous discussion. We will not delve into the definition of the V-

equivariant curvature of the normal bundle $\Omega_V^{\mathscr{N}\mathscr{M}_V}$ here and refer the readers to [5, 6] for instance. For our purposes it suffices to say that $\Omega_V^{\mathscr{N}\mathscr{M}_V}$ is the sum of two terms: the ordinary curvature Ω_∇ of the connection ∇, and the infinitesimal action of V on the normal bundle $L_V^{\mathscr{N}\mathscr{M}_V}$, which accounts for the equivariant action.

If the localisation locus $\mathscr{M}_V$ is a set of isolated fixed points $\{x_k\}_{k=1}^n$, the integral (2.35) collapses to a sum over the isolated fixed points, $i^*\alpha$ reduces to the 0-form $\alpha_0(x_k)$ evaluated at the fixed points, the curvature Ω_∇ vanishes, and the infinitesimal action of V on the normal bundle to x_k becomes the infinitesimal action $L_V(x_k)$ on the whole tangent bundle, of dimension 2ℓ. The Berline-Vergne formula (2.35) then reduces to

$$\int_{\mathscr{M}} \alpha = \int_{\mathscr{M}_V} \frac{i^*\alpha}{\mathrm{Pf}\left(-\frac{\Omega_\nabla + L_V^{\mathscr{N}\mathscr{M}_V}}{2\pi}\right)} = \sum_{x_k\in\mathscr{M}_V} \frac{\alpha_0(x_k)}{\prod_{i=1}^{\ell} \frac{\omega_{x_k,i}}{2\pi}} = (-2\pi)^\ell \sum_{x_k\in\mathscr{M}_V} \frac{\alpha_0(x_k)}{\mathrm{Pf}\, L_V(x_k)} \,, \tag{2.36}$$

reproducing the localisation formula (2.30).

3. Basics of supersymmetric localisation

The equivariant localisation formulae that we encountered in the previous section have become ubiquitous in Theoretical Physics, in particular in the context of supersymmetric systems. Prominent examples are the equivariant volumes of the moduli spaces of instantons [3, 13, 14] which appear in Nekrasov's microscopic derivation of the Seiberg-Witten solution of the low energy dynamics of $\mathscr{N}=2$ supersymmetric gauge theories in four dimensions, and the volumes of Sasakian manifolds [15] dual to the a-function in the context of supersymmetric AdS_5/CFT_4 dualities.

The main motivation for presenting equivariant localisation formulae for ordinary integrals here was that they serve as toy models for localisation formulae for path integrals of supersymmetric quantum field theories. Apart from the fact that, in contrast to the functional integrals of quantum field theory, finite-dimensional integrals are mathematically well-defined and equivariant localisation theorems can be proven rigorously, we will see that it is possible to draw a complete analogy between the notions that we have encountered in the context of equivariant localisation and those that we will encounter in the context of supersymmetric localisation. This correspondence is made explicit in Table 1.

Instead of the equivariant differential d_V which squares to the Lie derivative $-\mathscr{L}_V$, we have a conserved supercharge $\mathscr{Q}$ which squares to a bosonic charge B. Instead of even and odd polyforms, we have bosonic and fermionic fields. Instead of equivariantly closed polyforms, we have supersymmetric (or "BPS") observables, which are annihilated by the supercharge $\mathscr{Q}$ and therefore by B. Instead of computing integrals of equivariantly closed polyforms, we compute path integrals over a field space $\mathscr{F}$ with the insertion of supersymmetric observables $\mathscr{O}$. As we can deform the integral of an equivariantly closed polyform by inserting the exponential of an equivariantly exact equivariant polyform without changing the answer, we can deform the classical action in the path integral by adding a $\mathscr{Q}$-exact term, the supersymmetry variation of a fermionic operator invariant under the bosonic symmetry B. Instead of the zero locus $\mathscr{M}_V$ of the vector field V in the manifold $\mathscr{M}$, the localisation locus of the path integral over field space $\mathscr{F}$ is the BPS locus $\mathscr{F}_{\mathscr{Q}}$ of supersymmetric field configurations under the supercharge $\mathscr{Q}$. Finally, integrating out the bosonic and

Equivariant Localisation	**Supersymmetric Localisation**
$d_V = d - \iota_V$	$\mathscr{Q}$
$d_V^2 = -\mathscr{L}_V$	$\mathscr{Q}^2 = B$
Even/odd polyforms	Bosons/Fermions
$d_V \alpha = 0$	$\mathscr{Q}\mathscr{O} = 0$
$\int_{\mathscr{M}} \alpha = \int_{\mathscr{M}} \alpha \, e^{t d_V \beta}$,	$\int_{\mathscr{F}} [\mathscr{D}X] \; \mathscr{O} \; e^{-S[X]} = \int_{\mathscr{F}} [\mathscr{D}X] \; \mathscr{O} \; e^{-S[X] - t\mathscr{Q}\mathscr{P}_F[X]}$,
with $\mathscr{L}_V \beta = 0$	with $B\mathscr{P}_F[X] = 0$
$\mathscr{M}_V$	$\mathscr{F}_{\mathscr{Q}}$
Equiv. Euler class of $\mathscr{N}\mathscr{M}_V$	1-loop SDet

Table 1: Analogy between equivariant localisation for finite-dimensional integrals and supersymmetric localisation for path integrals.

fermionic field fluctuations transverse to the localisation locus in the limit where the localisation parameter $t \to \infty$ leads to a 1-loop super-determinant in the context of supersymmetric path integrals, analogous to the equivariant Euler class of the normal bundle to the localisation locus in the case of ordinary integrals.

I will exploit this analogy to present the salient aspects of supersymmetric localisation in this section. I start in section 3.1 by introducing path integrals of supersymmetric field theories, the cohomology of the supercharge $\mathscr{Q}$ and supersymmetric (or BPS, or $\mathscr{Q}$-closed) observables. In section 3.2 I will provide two reasons why path integrals of supersymmetric field theories with insertion of BPS observables localise to the locus $\mathscr{F}_{\mathscr{Q}}$ of supersymmetric field configurations. Finally, in section 3.3 I will schematically discuss the derivation of the localisation formula for the path integral of a supersymmetric field theory.

I refer the readers to [16] for a recent review which covers various aspects of supersymmetric localisation.

3.1 $\mathscr{Q}$-cohomology and path integrals of supersymmetric field theories

Let us consider a quantum field theory with a fermionic symmetry generated by a Grassmann odd charge $\mathscr{Q}$. We will be interested in supersymmetric field theories where $\mathscr{Q}$ is a supercharge.[5] The supercharge $\mathscr{Q}$ squares to a bosonic charge B

$$\mathscr{Q}^2 = B \; , \tag{3.1}$$

which may generate a linear combination of spacetime symmetries, global internal symmetries and (when acting on gauge-variant fields) gauge symmetries.

We will study BPS observables $\mathscr{O}_{BPS}$, gauge invariant operators which are preserved by the supercharge $\mathscr{Q}$:

$$\mathscr{Q}\mathscr{O}_{BPS} = 0 \; . \tag{3.2}$$

$\mathscr{O}_{BPS}$ may be a local operator, a product of local operators at separated points, or a non-local operator like a supersymmetric Wilson loop or 't Hooft loop or a surface operator. Our aim is to

[5] $\mathscr{Q}$ could also be a BRST charge. In that case the localisation argument that I will review in the following subsections realizes the gauge fixing.

compute the expectation value $\langle \mathscr{O}_{BPS} \rangle$ of the BPS observable exactly in the quantum theory:[6]

$$\langle \mathscr{O}_{BPS} \rangle = \int_{\mathscr{F}} [\mathscr{D}X] \; \mathscr{O}_{BPS} \, e^{-S[X]} \; . \tag{3.3}$$

First of all, an analogue of Stokes' theorem for path integrals of supersymmetric quantum field theories shows that the expectation value of a $\mathscr{Q}$-exact observable vanishes:

$$\langle \mathscr{Q}\mathscr{O} \rangle = \int_{\mathscr{F}} [\mathscr{D}X] \; (\mathscr{Q}\mathscr{O}) \, e^{-S[X]} = \int_{\mathscr{F}} [\mathscr{D}X] \; \mathscr{Q}(\mathscr{O} \, e^{-S[X]}) = 0 \; . \tag{3.4}$$

In the second equality I have used the invariance of the action under supersymmetry $\mathscr{Q}S[X] = 0$. We end up with an integral of a total derivative in field space, which vanishes provided that there are no boundary terms.[7] Whether or not this is the case depends on the asymptotics of the integrand in field space: I will assume that the integrand decays fast enough so that there are no boundary terms. This is typically the case.

As a result of (3.4), path integrals of supersymmetric field theories with insertions of $\mathscr{Q}$-closed observables only depend on the $\mathscr{Q}$-cohomology class of the inserted operators:

$$\langle \mathscr{O}_{BPS} \rangle = \langle \mathscr{O}_{BPS} + \mathscr{Q}\mathscr{O} \rangle \tag{3.5}$$

for any well-defined gauge invariant operator $\mathscr{O}$.

3.2 Supersymmetric path integrals localise

We will try to compute $\langle \mathscr{O}_{BPS} \rangle$ by localising the path integral of the quantum field theory to the BPS locus $\mathscr{M}_{\mathscr{Q}}$ of $\mathscr{Q}$-supersymmetric field configurations. In our discussion we require that the path integral is well-defined, in particular that it is free of infrared divergences. This can be achieved by placing the supersymmetric quantum field theory on a compact manifold or in an Omega background [3, 14]. Defining supersymmetric field theories on curved manifolds will be the purpose of the next section.

Similarly to what we saw in section 2.3 in the case of finite-dimensional integrals of equivariantly closed forms, there are two ways to see that path integrals of supersymmetric field theories localise to the locus $\mathscr{F}_{\mathscr{Q}}$ of BPS field configurations which are annihilated by the supercharge $\mathscr{Q}$.

1st localisation argument This argument is due to Witten [17] and is analogous to the argument based on the Poincaré lemma and Stokes' theorem presented in section 2.3. It goes as follows. Consider a quantum field theory with fields collectively called X, defined on a field space $\mathscr{F}$ over which we path integrate. Assume that there is a symmetry group G which acts freely on field space $\mathscr{F}$ and consider an operator $\mathscr{O}$ invariant under G. Then we can introduce collective coordinates for the G-action and integrate over them to get the volume of the group G. This volume factor multiplies a left-over path integral over $\mathscr{F}/G$, the space of orbits of G in field space:

$$\int_{\mathscr{F}} [\mathscr{D}X] \; \mathscr{O} \, e^{-S[X]} = \mathrm{Vol}(G) \cdot \int_{\mathscr{F}/G} [\mathscr{D}X] \; \mathscr{O} \, e^{-S[X]} \; . \tag{3.6}$$

[6] To be precise I should divide by normalization factors $\int_{\mathscr{F}} [\mathscr{D}X] \, e^{-S[X]}$ in (3.3). I prefer not to clutter formulae with these normalisation factors, that can be reinstated without affecting the following derivation.

[7] By assumption $\mathscr{Q}$ is not anomalous, so the integration measure is $\mathscr{Q}$-invariant.

If the global symmetry group G is generated by a fermionic charge $\mathscr{Q}$, the associated collective coordinate θ is a Grassmann variable and the volume vanishes by the rules of Berezin integrals: $\mathrm{Vol}(G) = \int d\theta\ 1 = 0$. Of course a supercharge $\mathscr{Q}$ cannot act freely on the whole field space $\mathscr{F}$, otherwise (3.6) would vanish even for the identity operator and we would not be able to normalise correlators. The supercharge $\mathscr{Q}$ has fixed points, which form the *BPS locus* of (bosonic) $\mathscr{Q}$-invariant field configurations

$$\mathscr{F}_{\mathscr{Q}} = \{[X] \in \mathscr{F} \mid \text{fermions} = 0,\ \mathscr{Q}(\text{fermions}) = 0\} . \tag{3.7}$$

The supercharge $\mathscr{Q}$ acts freely on the complement of the BPS locus, $\mathscr{F} \setminus \mathscr{F}_{\mathscr{Q}}$, and we can apply the above argument there to learn that the path integral with insertions of $\mathscr{Q}$-closed observables vanishes over $\mathscr{F} \setminus \mathscr{F}_{\mathscr{Q}}$. Therefore we conclude that the path integral over field space $\mathscr{F}$ localises to a subspace, the BPS locus $\mathscr{F}_{\mathscr{Q}}$ of $\mathscr{Q}$-invariant field configurations.

2nd localisation argument In the second localisation argument we use the freedom to deform the path integrand of a supersymmetric quantum field theory by a $\mathscr{Q}$-exact term to force the path integral to localise to the BPS locus $\mathscr{F}_{\mathscr{Q}}$ [1, 17, 4].

We wish to compute the expectation value (3.3) of a BPS observable (3.2). Since we have shown in (3.5) that the expectation value $\langle \mathscr{O}_{BPS} \rangle$ only depends on the $\mathscr{Q}$-cohomology class $[\mathscr{O}_{BPS}]$, we may consider a $\mathscr{Q}$-cohomologous representative which is obtained by adding to the classical action the $\mathscr{Q}$-variation of a B-invariant fermionic functional:

$$\langle \mathscr{O}_{BPS} \rangle = \int_{\mathscr{F}} [\mathscr{D}X]\ \mathscr{O}_{BPS}\ e^{-S[X] - t\mathscr{Q}\mathscr{P}_F[X]} \qquad \forall\, t . \tag{3.8}$$

The B-invariance of $\mathscr{P}_F$ ensures that the deformed observable is $\mathscr{Q}$-cohomologous to the original observable: $[\mathscr{O}_{BPS}\ e^{-t\mathscr{Q}\mathscr{P}_F[X]}] = [\mathscr{O}_{BPS}]$. The equality (3.8) is valid for all t and all fermionic functionals $\mathscr{P}_F[X]$ invariant under B that do not change the asymptotics of the integrand. Then it is straightforward to see that the derivative of the RHS of (3.8) is t-independent because of the analogue of Stokes' theorem (3.4):

$$\begin{aligned} \frac{d}{dt} \int_{\mathscr{F}} [\mathscr{D}X]\ \mathscr{O}_{BPS}\ e^{-S[X]-t\mathscr{Q}\mathscr{P}_F[X]} &= -\int_{\mathscr{F}} [\mathscr{D}X]\ (\mathscr{Q}\mathscr{P}_F) \mathscr{O}_{BPS}\ e^{-S[X]-t\mathscr{Q}\mathscr{P}_F[X]} = \\ &= -\int_{\mathscr{F}} [\mathscr{D}X]\ \mathscr{Q}\left(\mathscr{P}_F \mathscr{O}_{BPS}\ e^{-S[X]-t\mathscr{Q}\mathscr{P}_F[X]}\right) = 0 . \end{aligned} \tag{3.9}$$

In practice we will assume that the bosonic part of the deformation term $\mathscr{Q}\mathscr{P}_F[X]|_{bos}$ is positive semi-definite and consider a non-negative localisation parameter t. We can then evaluate $\langle \mathscr{O}_{BPS} \rangle$ by taking $t \to +\infty$ in the RHS of (3.8):

$$\langle \mathscr{O}_{BPS} \rangle = \lim_{t \to +\infty} \int_{\mathscr{F}} [\mathscr{D}X]\ \mathscr{O}_{BPS}\ e^{-S[X]-t\mathscr{Q}\mathscr{P}_F[X]} . \tag{3.10}$$

In this limit the integrand is dominated by the saddle points of the *localising action*

$$S_{loc}[X] := \mathscr{Q}\mathscr{P}_F[X] . \tag{3.11}$$

If we choose the localising Lagrangian density

$$\mathscr{L}_{loc} = \mathscr{Q} \sum_{\psi_X} \left((\mathscr{Q}\psi_X)^\dagger \psi_X + \psi_X^\dagger (\mathscr{Q}\psi_X^\dagger)^\dagger \right) , \tag{3.12}$$

where ψ_X runs over all fermionic fields in the theory, and whose bosonic part is a sum of squares of supersymmetry variations

$$\mathscr{L}_{loc}|_{bos} = \sum_{\psi_X} \left(|\mathscr{Q}\psi_X|^2 + |\mathscr{Q}\psi_X^\dagger|^2 \right) , \tag{3.13}$$

then the saddle points of the localising action S_{loc}, to which the path integral localises to, are nothing but the BPS configurations

$$\psi_X = \psi_X^\dagger = 0 , \qquad \mathscr{Q}\psi_X = \mathscr{Q}\psi_X^\dagger = 0 \tag{3.14}$$

that belong to the BPS locus $\mathscr{F}_\mathscr{Q}$ (3.7). In the following I will collectively denote by X_0 the saddle point configurations of S_{loc}: $X_0 \in \mathscr{F}_\mathscr{Q}$.

3.3 Localisation formula for supersymmetric path integrals

To evaluate the deformed path integral (3.10), let us expand the fields X about the saddle point configurations of S_{loc},

$$X = X_0 + \frac{1}{\sqrt{t}}\, \delta X \tag{3.15}$$

and take the limit $t \to \infty$. The inverse of the localisation parameter t behaves in (3.10) as an auxiliary Planck constant $\hbar_{aux} = 1/t$. The semiclassical loop expansion of the action in $\hbar_{aux} = 1/t$ reduces in the limit to

$$S[X_0] + \frac{1}{2} \iint \frac{\delta^2 S_{loc}[X]}{\delta X^2}\bigg|_{X=X_0} (\delta X)^2 . \tag{3.16}$$

This result is "1-loop exact": higher orders in the functional Taylor expansion are weighted by negative powers of t and vanish in the $t \to \infty$ limit. Integrating out the fluctuations δX normal to the localisation locus $\mathscr{F}_\mathscr{Q}$ at 1-loop, we obtain schematically the localisation formula

$$\langle \mathscr{O}_{BPS} \rangle = \int_{\mathscr{F}_\mathscr{Q}} [\mathscr{D}X_0]\ \mathscr{O}_{BPS}|_{X=X_0}\ e^{-S[X_0]} \frac{1}{\mathrm{SDet}\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right]} . \tag{3.17}$$

The original path integral (3.3) over field space $\mathscr{F}$ has localised to a lower-dimensional integral over the BPS locus $\mathscr{F}_\mathscr{Q}$, where the classical action evaluated on the BPS locus $S[X_0]$ is corrected due to integrating out the field fluctuations δX normal to $\mathscr{F}_\mathscr{Q}$. This leads to a 1-loop Super-Determinant of the operator $\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}$, the ratio of the determinants of the operators appearing at quadratic orders in the bosonic and fermionic fluctuations in (3.16) respectively.

I would like to stress again that the localisation formula (3.17) is an example of exact semiclassical approximation with respect not to the quantum parameter $\hbar$ of the original action $S[X]$ (which was set to 1 throughout the section), but rather to an auxiliary quantum parameter $\hbar_{aux} = 1/t$. The original action $S[X]$ weighted by $1/\hbar$ is a spectator in the computation.

The localisation formula (3.17) is a very powerful simplification because it reduces the dimensionality of the path integral that one needs to compute to evaluate expectation values and correlators of $\mathscr{Q}$-invariant observables. Depending on the spacetime dependence of the field configurations belonging to the localisation locus $\mathscr{F}_\mathscr{Q}$, one may be left with the path integral of a lower-dimensional quantum field theory [18] or, in favourable cases where the localisation locus consists of constant field configurations, with a finite-dimensional integral of a 0-dimensional quantum field theory such as a matrix model [4, 19].

Finally, note that, exactly as in the equivariant localisation theorems, there is some freedom in deriving supersymmetric localisation formulae. First of all, if there are multiple conserved supercharges $\mathscr{Q}_1, \dots, \mathscr{Q}_r$, any of them can be used to define supersymmetric observables and run the localisation argument. Secondly, at fixed localising supercharge $\mathscr{Q}$ and therefore fixed set of BPS observables, we can still work in different localisation schemes depending on the choice of $\mathscr{P}_F$ in (3.10) and (3.11), which need not be the canonical (3.12). These choices affect the localisation locus, which is the set of saddle points of the localising action $\mathscr{Q}\mathscr{P}_F$, hence the 1-loop determinant in (2.30). So different localisation schemes typically lead to different-looking forms of the localisation formula (2.30). The answers in different localisation schemes must eventually agree, because they differ by integrals of $\mathscr{Q}$-exact observables, which vanish by (3.4). This observation has been exploited recently in the so-called Higgs branch localisation [20, 21, 22, 23, 24], alternative to the standard Coulomb branch localisation.

In this section I have explained how a localisation argument similar to the one in equivariant cohomology can be formally applied to path integrals of supersymmetric quantum field theories. The path integral over the whole field space can be exactly reduced to a (path or ordinary) integral over a localisation locus of $\mathscr{Q}$-supersymmetric field configurations, weighted by the classical action and a 1-loop superdeterminant of the fluctuations transverse to the localisation locus (3.17). Following [4, 19], this method has been applied extensively to rigid supersymmetric field theories formulated on curved compact manifolds, whose path integrals are free of infrared divergences. In the next section I will review the formulation of rigid supersymmetric field theories in curved space and then apply it to $3d$ $\mathscr{N}=2$ supersymmetric theories on the round 3-sphere. In the final section I will apply the localisation argument introduced in this section to such theories.

4. Rigid supersymmetry in curved space

Let us consider a supersymmetric field theory in flat space. By this we mean a field theory in flat space endowed with a supersymmetry algebra generated by infinitesimal supersymmetry transformations $\delta^{(0)}$ which are realised on the component fields of the supermultiplets in the theory

$$\begin{aligned} \delta^{(0)}(\text{boson}) &= (\text{fermion}) \\ \delta^{(0)}(\text{fermion}) &= (\text{boson}) \ , \end{aligned} \tag{4.1}$$

with (schematically) $(\delta^{(0)})^2 \sim \gamma^\mu \partial_\mu$, and a supersymmetric action $S^{(0)} = \int \mathscr{L}^{(0)}$, integral of a Lagrangian density $\mathscr{L}^{(0)}$ whose supersymmetry variation is a total derivative:

$$\delta^{(0)} \mathscr{L}^{(0)} = \partial_\mu (\dots)^\mu \ . \tag{4.2}$$

The question that we want to answer in this lecture is whether this structure survives when the field theory is defined on a curved manifold M. We will see that the answer depends on the kind of supersymmetry algebra, or equivalently on the supermultiplet to which the supercurrent belongs, and on properties of the manifold M, such as the existence of spinor supersymmetry parameters satisfying certain differential equations.

Historically two strategies have been followed to define supersymmetry in curved space: a first strategy based on correcting flat space supersymmetry by trial and error, that I will review in section 4.1, and a second more systematic strategy based on non-linearly coupling the supersymmetric field theory to supergravity and then taking a rigid limit which makes supergravity non-dynamical, which I will review in sections 4.2 and 4.3. Finally in section 5.1 I will review the coupling to background supergravity for $3d$ $\mathcal{N}=2$ supersymmetric theories with an R-symmetry and derive the conditions for having supersymmetric backgrounds.

More details on the content of this section can be found in [25, 26].

4.1 Supersymmetry in curved space from trial and error

How do we define a supersymmetric field theory in curved space? As a zeroth order guess, we covariantise the flat space supersymmetry transformations $\delta^{(0)}$ and supersymmetric Lagrangian $\mathscr{L}^{(0)}$, replacing the flat metric by the curved metric and ordinary derivatives by covariant derivatives. While this procedure defines a field theory on curved space, it does not define a supersymmetric one, because $\delta^{(0)}\big|_{\substack{\eta\to g\\ \partial\to\nabla}}\mathscr{L}^{(0)}\big|_{\substack{\eta\to g\\ \partial\to\nabla}} \neq \nabla_\mu(\dots)^\mu$, where ∇_μ is the covariant derivative. We wish to correct this zeroth order guess in an expansion in inverse powers of some characteristic curvature length r of the curved manifold M,[8]

$$\begin{aligned}\delta &= \delta^{(0)}\big|_{\substack{\eta\to g\\ \partial\to\nabla}} + \sum_{n\geq 1}\frac{1}{r^n}\delta^{(n)}\\ \mathscr{L} &= \mathscr{L}^{(0)}\big|_{\substack{\eta\to g\\ \partial\to\nabla}} + \sum_{n\geq 1}\frac{1}{r^n}\mathscr{L}^{(n)}\ .\end{aligned} \tag{4.3}$$

Order by order in the $1/r$ expansion, we have to ensure that the supersymmetry algebra closes and the Lagrangian is supersymmetric. The expansion terminates if we can close the supersymmetry algebra exactly as a function of r and ensure that the supersymmetry variation of the Lagrangian is a total derivative:

$$\delta\mathscr{L} = \nabla_\mu(\dots)^\mu\ . \tag{4.4}$$

This strategy has been used extensively in the early literature, but it has a few drawbacks: it has to be done case by case, depending on the curved Riemannian manifold M, it is painful, and there is no guarantee that it will work. A priori, the $1/r$ expansion is an infinite Taylor expansion. However, r carries inverse mass dimension so it is clear on dimensional grounds that the expansion has to terminate:[9] the correction to the flat space supersymmetry and Lagrangian are due to relevant operators, whose effect is negligible in the ultraviolet, and there are finitely many of them. So once we have run out of relevant operators in the expansion (4.3), either the supersymmetry algebra closes and the Lagrangian is supersymmetric, in which case we are in business and we have defined a supersymmetric field theory on the curved space M, or it does not, and we have no supersymmetric theory on M.

[8] If M is compact, we may think of r as the radius associated to constant rescalings of the metric.

[9] I thank Zohar Komargodski for teaching me this argument.

Whenever it has been possible to define a supersymmetric field theory on curved space, it turns out that the Taylor expansion in $1/r$ for the supersymmetry transformation δ terminates at first order, while the Taylor expansion for the Lagrangian $\mathscr{L}$ terminates at second order. This fact is nicely explained in an alternative approach to define rigid supersymmetry in curved space which was proposed by Festuccia and Seiberg [25] building on [27, 28] and to which we now turn.

4.2 Supersymmetry on curved space from background supergravity

It is well known that a field theory defined on a curved spacetime can be formulated in a two-step procedure. First, we couple the flat space field theory to gravity, so that the metric is allowed to fluctuate. Then we take a *rigid limit* where Newton's constant $G_N \to 0$, decoupling gravity, but at the same time sending the metric to a fixed curved metric on the spacetime M, rather than the original flat metric. As a result, we are left with a field theory coupled to a background nontrivial metric on the curved spacetime M.

The logic of Festuccia and Seiberg is analogous. Since the aim is to define a *supersymmetric* field theory on curved space, we should first couple the supersymmetric field theory to supergravity. The supergravity multiplet typically includes the metric $g_{\mu\nu}$, its superpartner the gravitino $\psi_{\mu\alpha}$, and auxiliary fields. Then we take a *rigid limit* where Newton's constant $G_N \to 0$, the metric is sent to a fixed background metric, and the auxiliary fields (scaled by appropriate powers of G_N according to their dimension) are also sent to fixed backgrounds.

Since in the rigid limit the gravity multiplet is not dynamical, we do not need to solve the equations of motion for its components: the background gravity multiplet is off-shell. In particular the auxiliary fields are not determined in terms of the other components of the multiplet. We only require that the background for the gravity multiplet, which is bosonic, is supersymmetric. This is achieved by imposing that the supersymmetry variations of the gravitini (and in general all fermions in the gravity multiplet) vanish in the rigid limit:

$$\psi_{\mu\alpha} = 0 \ , \qquad \delta\psi_{\mu\alpha} = 0 \ . \tag{4.5}$$

This leads to generalised Killing spinor equations,[10] a set of first order differential equations which are the local conditions for preserving supersymmetry on the curved space M. These equations only involve the background bosonic fields in the gravity multiplet and are insensitive to most details of the field theory. They only depend on the flat space supersymmetry algebra, which determines the supermultiplet in which the supercurrent associated to supersymmetry transformations transforms in, which in turn determines a conjugate supergravity multiplet that it couples to, and not on the specific field content of the field theory.

Once equipped with a solution of the Killing spinor, we can find the supersymmetry transformations of the fields X_{SFT} and the Lagrangian $\mathscr{L}_{SFT}$ of the supersymmetric field theory on curved space M from the local supersymmetry transformations and Lagrangian in the coupled field theory-supergravity, by taking the rigid limit and plugging in a background for the gravity multiplet which solves the Killing spinor equations (4.5):

$$\begin{aligned} \delta_{sugra} X_{SFT} &\longrightarrow \delta X_{SFT} \\ \mathscr{L}_{SFT+SuGra} &\longrightarrow \mathscr{L}_{SFT} \ . \end{aligned} \tag{4.6}$$

[10] We will loosely refer to these equations simply as Killing spinor equations in the following.

By construction the rigid supersymmetry algebra on curved space closes, being a subalgebra of the local supersymmetry algebra of supergravity, and the supersymmetry variation of the Lagrangian is a total derivative, as it was before the limit.

This approach is very useful because it provides a global view on the space of supersymmetric field theories on curved space. First of all, the Killing spinor equations (4.5) can be used to classify all curved space supersymmetric backgrounds for a given supergravity multiplet, up to global issues. Secondly, one can derive formulae for the supersymmetry transformations and supersymmetric Lagrangians which are valid for any solution of the Killing spinor equations, with no need of specifying the particular solution. Finally, as noted in [25], the $1/r$ expansion in (4.3) can be reinterpreted as an expansion in the auxiliary fields of the gravity multiplet. Auxiliary fields only appear linearly in the local supersymmetry transformations and quadratically in the coupled field-theory supergravity Lagrangians, thus explaining why the expansions in (4.3) stop at orders $n = 1$ and $n = 2$ respectively.

4.3 Coupling to background fields

The procedure developed by Festuccia and Seiberg in [25] and outlined above is straightforward to apply provided that we know the supergravity theory that couples to the supersymmetric field theory for the matter fields is known: in this case we only need to take a limit. If the relevant supergravity theory is not known, one can still reconstruct the Killing spinor equations and the couplings to supergravity. Since we are interested in taking the rigid limit to define supersymmetry on curved space, it suffices to keep the supergravity multiplet as a non-dynamical background. So we need to couple the dynamical supersymmetric field theory to a background supergravity, at the non-linear level. In this and the next section we will review how this can be done, following [26].

Recall the concept of minimal coupling, which enables us to couple a conserved current to a conjugate gauge field. Let us consider a field theory with fields collectively denoted as Φ and Lagrangian density $\mathscr{L}^{(0)}(\Phi,\partial\Phi)$. If the field theory enjoys an internal symmetry $\Phi \to e^{i\alpha}\Phi$ with parameter α, there is an associated conserved Noether current

$$j^{(0)\mu} \equiv \frac{\delta\phi}{\delta\alpha}\cdot\Pi^\mu = \frac{\delta\phi}{\delta\alpha}\cdot\frac{\partial\mathscr{L}^{(0)}}{\partial\partial_\mu\Phi} \qquad s.t. \qquad \partial_\mu j^{(0)\mu} = 0\,, \tag{4.7}$$

which defines the conserved charge $Q = \int j_0^{(0)}$ that generates the infinitesimal internal symmetries.

If we want to make the symmetry local, we minimally couple the theory to a gauge field A_μ conjugate to the conserved current $j^{(0)\mu}$ by promoting ordinary derivatives to gauge covariant derivatives:

$$\mathscr{L}^{(0)}(\Phi,\partial\Phi) \to \mathscr{L} \equiv \mathscr{L}^{(0)}(\Phi,D\Phi)\,, \qquad D_\mu\Phi = (\partial_\mu - iA_\mu)\Phi\,, \tag{4.8}$$

where A_μ acts on Φ in the appropriate representation. A_μ has a gauge symmetry reflecting the conservation of the current. Then the conserved current of the uncoupled theory can be written as

$$j^{(0)\mu} = -\frac{\partial\mathscr{L}}{\partial A_\mu}\bigg|_{A=0} = -\frac{\delta S}{\delta A_\mu}\bigg|_{A=0} \tag{4.9}$$

and the coupled Lagrangian can be written as

$$\mathscr{L} = \mathscr{L}^{(0)}(\Phi,\partial\Phi) - j^{(0)\mu}A_\mu + \mathscr{O}(A_\mu A^\mu)\,, \tag{4.10}$$

where the non-linear "seagull" terms $\mathscr{O}(A_\mu A^\mu)$ can be fixed by requiring gauge invariance of $\mathscr{L}$. The gauge field A_μ can be made dynamical by adding a Yang-Mills type kinetic Lagrangian $\mathscr{L}_{kin}(A, \partial A) \sim \frac{1}{g^2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$, or can be kept as an external background gauge field. We will be interested in the latter case.

Similarly we deal with spacetime symmetries. Translation symmetries are generated by the momentum vector P^μ. The associated currents in a Poincaré invariant theory are a conserved symmetric energy-momentum tensor $T^{(0)\mu\nu} = T^{(0)\nu\mu}$: $\partial_\mu T^{(0)\mu\nu} = 0$. Minimal coupling is achieved by replacing the flat metric $\eta_{\mu\nu}$ by a curved metric $g_{\mu\nu} = \eta_{\mu\nu} - 2h_{\mu\nu}$, and ordinary derivatives ∂_μ by general covariant derivatives ∇_μ in the Lagrangian,

$$\mathscr{L}^{(0)} \to \mathscr{L} \equiv \mathscr{L}^{(0)}\Big|_{\substack{\eta \to g \\ \partial \to \nabla}} \tag{4.11}$$

and defining the action as $S = \int \sqrt{g}\, \mathscr{L}$. Given the action S resulting from minimal coupling, the energy-momentum tensor is computed as its first functional derivative with respect to the metric

$$T^{(0)\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}\Big|_{g=\eta} , \tag{4.12}$$

so the energy-momentum tensor is the conjugate variable to the graviton $h_{\mu\nu}$. The Lagrangian of the field theory in curved space can be written to linear order as

$$\mathscr{L} = \mathscr{L}^{(0)} - T^{(0)\mu\nu} h_{\mu\nu} + \dots , \tag{4.13}$$

where the dots denote non-linear terms which again are constrained by general coordinate invariance.

If our purpose is to couple the field theory to gravity the metric can be made dynamical by introducing a kinetic action such as the Einstein-Hilbert action $\mathscr{L}_{EH}(g, \partial g) \sim \frac{1}{G_N} \int \sqrt{g}\, R$. Then we can obtain the field theory on a curved space M by taking a rigid limit where $G_N \to 0$, making gravity non-dynamical, and the metric approaches a fixed background $g_{\mu\nu}$. For the purpose of defining a field theory on curved space, it suffices to couple the field theory to this fixed background metric $g_{\mu\nu}$ non-linearly.

Note that while the procedure of *minimal* coupling is unique, there are ambiguities in coupling the flat space field theory to the curved background. We could for instance add couplings proportional to the curvature, such as conformal mass terms $R\phi^2$, which vanish in the flat space limit.

If the field theory is supersymmetric, it possesses at least one conserved Grassmann spinorial supercharge Q_α (and the conjugate supercharge). The associated Noether current $S_\alpha^{(0)\mu}$ is called the *supercurrent*: $\partial_\mu S_\alpha^{(0)\mu} = 0$. The conjugate fermionic gauge field is the gravitino $\psi_{\mu\alpha}$. The linearised coupling of the supersymmetric field theory to the gravitino is via

$$\delta \mathscr{L}^{lin} \supset -\frac{1}{2} S^{(0)\mu} \psi_\mu + c.c. \tag{4.14}$$

The gravitino can be made dynamical by introducing a kinetic action such as $\mathscr{L}_{kin}(\psi, \partial\psi) \sim \frac{1}{G_N} \int \sqrt{g}\, \overline{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho$. As we are interested in the rigid supersymmetric field theory obtained in the $G_N \to 0$ limit, we will keep the gravitino non-dynamical.

Because supersymmetries square to translations in the flat space supersymmetry algebra, the supercurrent and the energy-momentum tensor reside in a common supermultiplet, called the *supercurrent multiplet*. Their conjugate variables, the gravitino and the graviton, sit in a conjugate supermultiplet called the *(super)gravity multiplet*:

$$\begin{array}{c|ccc} \text{Supercurrent multiplet} & T^{\mu\nu} & S^{\mu}_{\alpha} & \dots \\ \hline \text{(Super)Gravity multiplet} & h_{\mu\nu} & \psi_{\mu\alpha} & \dots \end{array} \tag{4.15}$$

The dots above denote additional fields belonging to the supermultiplets, which depend on the particular supersymmetry algebra under consideration. The linearised supersymmetric coupling in the Lagrangian takes the form

$$\delta\mathscr{L}^{lin} \supset -T^{\mu\nu}h_{\mu\nu} - \frac{1}{2}(S^{(0)\mu}\psi_{\mu} + c.c.) + \dots \tag{4.16}$$

where the ellipses involve extra fields in the above multiplets that we are not specifying yet.

Several supercurrent multiplets have been studied in the literature (see [28, 29] for an overview). They differ in the superpartners of the energy-momentum tensor and the supercurrent which fill the dots in (4.15)-(4.16) and lead to different supersymmetry algebras for the associated conserved Noether charges. Each of these supercurrent multiplets is paired with a conjugate supergravity multiplet, which differ in their auxiliary field content. For instance, every supersymmetric field theory has a supercurrent multiplet called the $\mathscr{S}$-multiplet. Supersymmetric field theories without Fayet-Iliopoulos parameters and with a trivial Kähler form on their moduli space have a Ferrara-Zumino supercurrent multiplet. Supersymmetric field theories with a $U(1)$ R-symmetry have a supercurrent $\mathscr{R}$-multiplet. Superconformal field theories have a superconformal supercurrent multiplet.

For the purpose of localisation, we will require that our supersymmetric field theories preserve four supercharges (such as $\mathscr{N}=1$ supersymmetry in four dimensions and dimensional reductions thereof) and a $U(1)_R$ symmetry. We will work therefore with the supercurrent $\mathscr{R}$-multiplet of [29]. For definiteness and to make connection with Antonio Amariti's lectures in the same school, from now on we will focus on three-dimensional field theories with $\mathscr{N}=2$ supersymmetry and a $U(1)_R$ symmetry. All the computations that follow can be generalised to other dimensions.

4.4 Coupling the 3d $\mathscr{R}$-multiplet to background supergravity

Let us start by discussing the flat space supersymmetry algebra in three dimensions. I will work in Euclidean signature to define the path integral. In Euclidean signature, spinors and scalars which are conjugate in Minkowskian signature are complexified and independent.[11] I will denote the remnant of Minkowskian signature complex conjugation by a tilde. In particular the supercharges which are conjugate in Lorentzian signature will be denoted as Q and $\widetilde{Q}$. In this section I follow very closely [26], to which I refer the readers for more details and conventions.

[11]However one needs to specify reality conditions for the fields in defining the path integration contour.

The $\mathscr{N}=2$ supersymmetry algebra on flat $\mathbb{R}^3$ with $U(1)_R$ symmetry is

$$\{Q_\alpha,\widetilde{Q}_\beta\}=2\gamma^\mu_{\alpha\beta}P_\mu+2i\varepsilon_{\alpha\beta}Z \tag{4.17}$$

$$\{Q_\alpha,Q_\beta\}=0 \qquad \{\widetilde{Q}_\alpha,\widetilde{Q}_\beta\}=0 \tag{4.18}$$

$$[R,Q_\alpha]=-Q_\alpha \qquad [R,\widetilde{Q}_\alpha]=+\widetilde{Q}_\alpha \tag{4.19}$$

$$[Z,Q_\alpha]=[Z,\widetilde{Q}_\alpha]=[Z,R]=0 \tag{4.20}$$

Here the gamma matrices γ^μ are Pauli matrices, and ε is the antisymmetric tensor used to lower/raise $SU(2)$ doublet spinor indices. Z is the real central charge of the $\mathscr{N}=2$ supersymmetry algebra, which provides a BPS bound for the mass $M\geq|Z|$. The 3d $\mathscr{N}=2$ supersymmetry algebra can be obtained by dimensional reduction of the 4d $\mathscr{N}=1$ supersymmetry algebra: in the reduction the Kaluza-Klein momentum P_4 becomes the central charge Z. R is the R-charge generating $U(1)_R$ symmetry transformations. The supercharges Q and $\widetilde{Q}$ lower and raise the R-charge by one unit.

The conserved currents associated to the charges which generate the supersymmetry algebra (4.17)-(4.20) reside in the supercurrent $\mathscr{R}$-multiplet [29]

$$\mathscr{R}^\mu=(T^{\mu\nu},S^\mu_\alpha,\widetilde{S}^\mu_\alpha,j^{(R)\mu},j^{(Z)\mu},J^{(Z)}) \tag{4.21}$$

which comprises the energy-momentum tensor $T^{\mu\nu}$ (for the momentum P^μ), the supercurrents S^μ_α, $\widetilde{S}^\mu_\alpha$ (for the supercharges Q_α, $\widetilde{Q}_\alpha$), the R-current $j^{(R)\mu}$ (for the R-charge R), the Z-current $j^{(Z)\mu}$ (for the central charge Z), and finally a topologically conserved string current $i\varepsilon^{\mu\nu\rho}\partial_\rho J^{(Z)}$.

The conjugate *new minimal* supergravity multiplet $\mathscr{H}$

$$\mathscr{H}_\mu=(h_{\mu\nu},\psi_{\mu\alpha},\widetilde{\psi}_{\mu\alpha},A_\mu,C_\mu,B_{\mu\nu}) \tag{4.22}$$

comprises the graviton $h_{\mu\nu}$, the gravitini $\psi_{\mu\alpha}$, $\widetilde{\psi}_{\mu\alpha}$, two gauge 1-forms A_μ and C_μ, and a gauge 2-form $B_{\mu\nu}$. C_μ is often dualised into a conserved vector $V^\mu=-i\varepsilon^{\mu\nu\rho}\partial_\mu C_\rho$, and the 2-form $B_{\mu\nu}$ is often traded for the scalar $H=\frac{i}{2}\varepsilon^{\mu\nu\rho}\partial_\mu B_{\nu\rho}$ which is Hodge dual to its field strength.

The supercurrent $\mathscr{R}$-multiplet and the conjugate new minimal supergravity $\mathscr{H}$-multiplet participate in a minimal coupling which to linear order reads

$$\delta\mathscr{L}^{lin}_{min}=-T^{\mu\nu}h_{\mu\nu}-\frac{1}{2}S^\mu\psi_\mu+\frac{1}{2}\widetilde{S}^\mu\widetilde{\psi}_\mu+j^{(R)\mu}(A_\mu-\frac{3}{2}V_\mu)+j^{(Z)\mu}C_\mu+J^{(Z)}H\ . \tag{4.23}$$

Upon integration by parts, the last term can also be written as $B\wedge dJ^{(Z)}$. Note that while the Z-current $j^{(Z)\mu}$ couples to the 1-form C_μ, the R-current $j^{(R)\mu}$ couples to $A_\mu-\frac{3}{2}V_\mu$ rather than to A_μ only.

Because the currents in (4.21) are conserved, the conjugate fields in (4.22) enjoy the gauge symmetries:

$$\delta h_{\mu\nu}=\partial_\mu\Lambda^{(h)}_\nu+\partial_\nu\Lambda^{(h)}_\mu \qquad \delta B_{\mu\nu}=\partial_\mu\Lambda^{(B)}_\nu-\partial_\nu\Lambda^{(B)}_\mu \tag{4.24}$$

$$\delta C_\mu=\partial_\mu\Lambda^{(C)} \qquad \delta A_\mu=\partial_\mu\Lambda^{(A)} \tag{4.25}$$

$$\delta\psi_{\mu\alpha}=\partial_\mu\varepsilon_\alpha \qquad \delta\widetilde{\psi}_{\mu\alpha}=\partial_\mu\widetilde{\varepsilon}_\alpha\ . \tag{4.26}$$

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The fields in (4.22) also transform under *constant* flat space supersymmetry transformations, with constant parameters ζ_α, $\widetilde{\zeta}_\alpha$. In particular, the supersymmetry transformations of the gravitini read

$$\delta_\zeta \psi_\mu = -i\varepsilon^{\nu\rho\lambda}\partial_\nu h_{\rho\mu}\gamma_\lambda\zeta - 2i(A_\mu - V_\mu)\zeta + H\gamma_\mu\zeta + \varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta + \partial_\mu(\dots) \tag{4.27}$$

$$\delta_{\widetilde{\zeta}}\widetilde{\psi}_\mu = -i\varepsilon^{\nu\rho\lambda}\partial_\nu h_{\rho\mu}\gamma_\lambda\widetilde{\zeta} + 2i(A_\mu - V_\mu)\zeta + H\gamma_\mu\widetilde{\zeta} - \varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta} + \partial_\mu(\dots)\,. \tag{4.28}$$

The total derivatives can be removed by gauge transformations (4.26).

So far we have considered a flat space supersymmetric field theory coupled to a fixed linearised background for the supergravity multiplet. Now we promote the supersymmetry parameters ζ, $\widetilde{\zeta}$ to local parameters and move on to linearised supergravity. The flat space supersymmetric Lagrangian $\mathscr{L}^{(0)}$ is invariant under constant supersymmetry transformations, but not under local ones. As is familiar in the derivation of Noether's theorem, the variation of $\mathscr{L}^{(0)}$ under local supersymmetry transformations is proportional to a derivative of the supersymmetry parameters, with the coefficient being the conserved supersymmetry current:

$$\delta_{\zeta,\widetilde{\zeta}}\,\mathscr{L}^{(0)} = S^\mu\partial_\mu\zeta - \widetilde{S}^\mu\partial_\mu\widetilde{\zeta}\,. \tag{4.29}$$

Comparing with the linearised minimal coupling (4.23), we see that the supersymmetry variation $\delta_{\zeta,\widetilde{\zeta}}\,\mathscr{L}^{(0)}$ can be absorbed by a gauge transformation of the gravitini (4.26), provided that the gauge parameters of the gravitini are identified with the local supersymmetry parameters as $\varepsilon = 2\zeta$, $\widetilde{\varepsilon} = 2\widetilde{\zeta}$.

Including the compensating gauge transformation with $\varepsilon = 2\zeta$, $\widetilde{\varepsilon} = 2\widetilde{\zeta}$, the local supersymmetry transformation of the gravitini can be written as

$$\begin{aligned}\delta_\zeta\psi_\mu &= 2\left(\partial_\mu - \frac{i}{2}\varepsilon^{\nu\rho\lambda}\partial_\nu h_{\rho\mu}\gamma_\lambda\right)\zeta - 2i(A_\mu - V_\mu)\zeta + H\gamma_\mu\zeta + \varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta\\ \delta_{\widetilde{\zeta}}\widetilde{\psi}_\mu &= 2\left(\partial_\mu - \frac{i}{2}\varepsilon^{\nu\rho\lambda}\partial_\nu h_{\rho\mu}\gamma_\lambda\right)\widetilde{\zeta} + 2i(A_\mu - V_\mu)\widetilde{\zeta} + H\gamma_\mu\widetilde{\zeta} - \varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta}.\end{aligned} \tag{4.30}$$

The differential operator in parenthesis above is nothing but the linearised version of the general covariant derivative acting on a spinor:

$$\nabla_\mu\chi = \left(\partial_\mu - \frac{i}{4}\omega_{\mu ab}\varepsilon^{abc}\gamma_c\right)\chi \tag{4.31}$$

where Greek (Roman) indices are curved (frame) indices, and $\omega_{\mu a}{}^b = e^b_\nu\nabla_\mu e^\nu_a$ is the spin connection.

The linearised local supersymmetry variations of the gravitini (4.30) can be easily promoted to non-linear local supersymmetry variations

$$\begin{aligned}\delta_\zeta\psi_\mu &= 2(\nabla_\mu - iA_\mu)\zeta + H\gamma_\mu\zeta + 2iV_\mu\zeta + \varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta + (\dots)\\ \delta_{\widetilde{\zeta}}\widetilde{\psi}_\mu &= 2(\nabla_\mu + iA_\mu)\widetilde{\zeta} + H\gamma_\mu\widetilde{\zeta} - 2iV_\mu\widetilde{\zeta} - \varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta} + (\dots)\,,\end{aligned} \tag{4.32}$$

where the ellipses stand for terms including the gravitini. Dimensional analysis arguments imply that no other non-linear terms appear in (4.32) in the rigid $G_N \to 0$ limit [26].

Equations (4.32) are the full supersymmetry variations of the gravitini in the non-linearly coupled field theory-supergravity theory, in the rigid limit in which the supergravity multiplet becomes non-dynamical. Imposing that the constant bosonic supergravity background is supersymmetric as in (4.5) leads to the Killing spinor equations

$$\begin{aligned}(\nabla_\mu - iA_\mu)\zeta &= -\frac{H}{2}\gamma_\mu\zeta - iV_\mu\zeta - \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta \\ (\nabla_\mu + iA_\mu)\widetilde{\zeta} &= -\frac{H}{2}\gamma_\mu\widetilde{\zeta} + iV_\mu\widetilde{\zeta} + \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta}\ .\end{aligned} \tag{4.33}$$

The signs in front of A_μ in the full covariant derivative of the spinors reflect the R-charges of the Killing spinors, namely $+1$ for ζ and -1 for $\widetilde{\zeta}$. C_μ does not appear because the Killing spinors do not carry central charge.

The generalised Killing spinor equations (4.33) are the main formulae of this section. We can define a rigid supersymmetric theory on a curved manifold M with given background metric $g_{\mu\nu}$, R-symmetry gauge field A_μ, scalar H and conserved vector V^μ if and only if there exists at least one non-vanishing spinor ζ or $\widetilde{\zeta}$ which solves the Killing spinor equations. To be precise, the analysis based on the Killing spinor equations is local on M and one should also worry about global constraints.

So the name of the game is to find background fields $g_{\mu\nu}$, A_μ, H and V^μ for a given manifold M, such that there exists at least one non-trivial solution to (4.33). The Killing spinor equations are linear in ζ, $\widetilde{\zeta}$, so solutions form a vector space, whose dimension is the number of supercharges preserved by the background. The maximal number of supercharges visible in this formalism is 4, the number of supercharges of $3d$ $\mathcal{N}=2$ supersymmetry in flat space.

Exercise 4.1. Consider the round 3-sphere of radius ℓ, with metric

$$\begin{aligned}ds^2 &= \ell^2(d\vartheta^2 + \cos^2\vartheta\ d\varphi_1^2 + \sin^2\vartheta\ d\varphi_2^2) \\ &= \frac{\ell^2}{4}(d\theta^2 + \sin^2\theta\ d\varphi^2 + (d\psi - \cos\theta\ d\varphi)^2)\ .\end{aligned} \tag{4.34}$$

The first parametrisation views S^3 as a T^2 fibration over an interval. The T^2 fibre coordinates are $\varphi_1 \sim \varphi_1 + 2\pi$ and $\varphi_2 \sim \varphi_2 + 2\pi$, whereas ϑ is the coordinate on the interval $[0, \frac{\pi}{2}]$. The two 1-cycles of the torus shrink at the opposite endoints of the interval, capping off the geometry. The second parametrisation views S^3 as a Hopf fibration over S^2: the base S^2 has spherical coordinates (θ, φ), the fibre has coordinate $\psi \sim \psi + 4\pi$ and a Dirac monopole connection. Check that the background given by the round metric (4.34) and the spherically symmetric fields

$$A_\mu = V_\mu = 0\ , \qquad H = -\frac{i}{\ell} \tag{4.35}$$

preserves 4 supercharges, corresponding to 2 Killing spinors ζ and 2 Killing spinors $\widetilde{\zeta}$ solving

$$\begin{aligned}\nabla_\mu\zeta &= \frac{i}{2\ell}\gamma_\mu\zeta \\ \nabla_\mu\widetilde{\zeta} &= \frac{i}{2\ell}\gamma_\mu\widetilde{\zeta}\ .\end{aligned} \tag{4.36}$$

So the round S^3 is a maximally supersymmetric background in $\mathcal{N}=2$ supersymmetry.

We started this subsection by discussing the $3d$ $\mathcal{N}=2$ supersymmetry algebra in flat space (4.17)-(4.20). Given some commuting Killing spinors ζ, η of R-charge 1 and $\widetilde{\zeta}$, $\widetilde{\eta}$ of R-charge -1, which solve the Killing spinor equations (4.33), we can find the curved space supersymmetry algebra by looking at the action of the supersymmetry transformations on generic fields.[12] Let $\phi_{(r,z)}$ be a field of R-charge r and Z-charge z. The curved space supersymmetry algebra reads

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\phi_{(r,z)} = -2i\left(\mathscr{L}'_K + \zeta\widetilde{\zeta}(z-rH)\right)\phi_{(r,z)} \tag{4.37}$$

$$\{\delta_\zeta, \delta_\eta\} = 0 \qquad\qquad \{\delta_{\widetilde{\zeta}}, \delta_{\widetilde{\eta}}\} = 0\ . \tag{4.38}$$

Here the spinor bilinear $K^\mu = \zeta\gamma^\mu\widetilde{\zeta}$ is a Killing vector by virtue of the Killing spinor equations (4.33). $\mathscr{L}'_K$ is a fully covariant Lie derivative along the Killing vector K

$$\mathscr{L}'_K\phi_{(r,z)} = \left[\mathscr{L}_K - iK^\mu\left(r(A_\mu - \frac{1}{2}V_\mu) + zC_\mu\right)\right]\phi_{(r,z)}\ , \tag{4.39}$$

with $\mathscr{L}_K$ the ordinary Lie derivative with respect to the vector field K.

The supersymmetry algebra (4.37)-(4.38) is the curved space counterpart of the flat space supersymmetry algebra (4.17)-(4.18). Instead of the translations generated by the momentum P_μ in flat space, in curved space we may have isometries of the background, generated by the Lie derivative $\mathscr{L}_K$, provided K^μ does not vanish. The Lie derivative term is corrected by the other background fields and becomes the covariant Lie derivative $\mathscr{L}'_K$ (4.39). Similarly the central charge term Z gets an extra contribution proportional to the R-charge in the curved background, leading to $\zeta\widetilde{\zeta}(z-rH)$. In backgrounds such that the tilded spinors are charge conjugate of the untilded one, H is purely imaginary (see for instance (4.35)). In those backgrounds $(z-rH)$ gives an imaginary part to the central charge, proportional to the R-charge. As first observed in [30] for S^3 and then explained in generality in [25], partition functions of supersymmetric quantum field theories are holomorphic in $(z-rH)$.

As in flat space, it is possible to define supermultiplets which are representations of the supersymmetry algebra (4.37)-(4.38), and use them to build supersymmetric actions. Again, a systematic method involves taking the rigid limit on a known coupled field theory-supergravity system as in (4.6). If the coupled system is not known, it can be worked out along the lines discussed in this section. The supersymmetry transformations of the field theory multiplets in the rigid limit can be obtained as we did for the gravitini. Similarly one can derive the supersymmetric actions. To get the curved space corrections to the flat space actions, one needs to non-linearly complete the minimal coupling (4.23), and express the components of the supercurrent multiplet in terms of the matter fields of the field theory. By construction the corrections to the flat space supersymmetry transformations and actions vanish in the flat limit in which the supergravity background becomes trivial. Because the local geometry about any regular point is flat, these corrections become smaller and smaller as we zoom about a point in space: the corrections to the flat space supersymmetric Lagrangian are given by relevant operators which are unimportant in the ultraviolet.

[12] A word of caution on a possible source of confusion. I take the Killing spinors to be commuting and the supersymmetry variations δ_ζ to be anticommuting. So here δ_ζ is analogous to acting with a supercharge Q in flat space. The supersymmetry parameters which multiply these Killing spinors are Grassmann valued.

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This analysis can be done once and for all, for all backgrounds which admit solutions to the Killing spinor equations. We will not list the curved space supersymmetry transformations and Lagrangians for $3d$ $\mathcal{N}=2$ field theory on a general curved supersymmetric background which admits solutions of (4.33). They can be found in the original reference [26]. To specialise the general curved space supersymmetry transformations and Lagrangians to a given supersymmetric background, we simply need to substitute the explicit expressions for the background supergravity fields and the Killing spinors that solve (4.33). In the next and final section, we will apply these formulae to the round S^3 background (4.34)-(4.35).

5. 3d $\mathcal{N}=2$ supersymmetry on S^3 and localisation

In this final section we will study $3d$ $\mathcal{N}=2$ supersymmetric gauge theories on the round 3-sphere, with background (4.34)-(4.35). I will introduce supersymmetry transformations of vector and chiral multiplets and their supersymmetric Lagrangians, which were originally derived by trial and error in [30, 31] generalising [19], and can also be obtained by plugging the supergravity background (4.34)-(4.35) in the general formulae of [26] mentioned at the end of the previous section. Finally I will apply the method of supersymmetric localisation introduced in section 3 and show that the exact path integrals of these three-dimensional supersymmetric quantum field theories can be reduced to ordinary integrals on localisation loci of constant field configurations [19, 30, 31], which is often possible to evaluate.

5.1 3d $\mathcal{N}=2$ supersymmetry on S^3

As we saw in exercise 4.1, the round S^3 background (4.34)-(4.35) preserves four supercharges, corresponding to two independent solutions ζ and two $\widetilde{\zeta}$ of the Killing spinor equations (4.36). Viewing S^3 as the $SU(2)$ group manifold, we can choose a frame with vielbein $e^i=\frac{\ell}{2}\mu^i$, $i=1,2,3$, where μ^i are the left-invariant Maurer-Cartan 1-forms of $SU(2)$:

$$\begin{aligned} e^1 &= \frac{\ell}{2}\left(\cos\psi d\theta-\sin\psi\sin\theta d\varphi\right)\\ e^2 &= \frac{\ell}{2}\left(\sin\psi d\theta+\cos\psi\sin\theta d\varphi\right)\\ e^3 &= \frac{\ell}{2}\left(d\psi-\cos\theta d\varphi\right)\,. \end{aligned} \tag{5.1}$$

The spin connection is $\omega^{ab}=-\frac{1}{\ell}\varepsilon^{abc}e^c$ and the covariant derivative of a spinor χ (4.31) is

$$\nabla_\mu\chi=\left(\partial_\mu+\frac{i}{2\ell}\gamma_\mu\right)\chi\,, \tag{5.2}$$

so the Killing spinor equations (4.36) are solved by constant spinors in this frame.

The supersymmetry algebra is

$$\begin{aligned} &\{\delta_\zeta,\delta_{\widetilde{\zeta}}\}\phi_{(r,z)}=-2i\left(\mathscr{L}_K+\zeta\widetilde{\zeta}(z+\frac{ir}{\ell})\right)\phi_{(r,z)}\\ &\{\delta_{\zeta_1},\delta_{\zeta_2}\}=0 \qquad\qquad \{\delta_{\widetilde{\zeta}_1},\delta_{\widetilde{\zeta}_2}\}=0\,, \end{aligned} \tag{5.3}$$

with $K^\mu = \zeta\gamma^\mu\widetilde{\zeta}$ a spinor bilinear Killing vector. The vector bilinears constructed from constant Killing spinors span the left-invariant Killing vectors of $SU(2)$.

Next I present supermultiplets, which form representations of the supersymmetry algebra (5.3). We are interested in supersymmetric gauge theories, so we will deal with vector multiplets in the gauge sector and with chiral multiplets in the matter sector. In flat space these multiplets are defined as in four-dimensional $\mathcal{N}=1$ supersymmetry, which leads to three-dimensional $\mathcal{N}=2$ supersymmetry upon dimensional reduction. Chiral multiplets are complex superfields Φ annihilated by the tilded supercovariant derivatives: $\widetilde{D}_\alpha\Phi = 0$. Vector multiplets are real superfields V defined modulo gauge transformations $e^V \to e^{i\widetilde{\Lambda}}e^V e^{-i\Lambda}$, where the parameter Λ is a chiral superfield in the adjoint representation of the gauge group, and $\widetilde{\Lambda}$ its conjugate antichiral superfield. It is customary to partially fix this gauge symmetry and go to the so-called Wess-Zumino gauge, where the residual gauge symmetry has an ordinary real gauge parameter.

In Wess-Zumino gauge, the *vector multiplet*

$$V = (A_\mu, \sigma, \lambda_\alpha, \widetilde{\lambda}_\alpha, D) \ , \tag{5.4}$$

comprises a gauge field A_μ plus a real scalar σ which descends from A_4 in four dimensions, the complex conjugate gaugini λ, $\widetilde{\lambda}$ and the real auxiliary fields D, all transforming in the adjoint representation of the gauge group. The vector multiplet has zero R-charge and central charge. The reality conditions mentioned above are to be thought of as the reality conditions in Minkowskian signature, but we will impose the same reality conditions to define the path integration contour for the field theory on S^3.

The supersymmetry transformations $\delta = \delta_\zeta + \delta_{\widetilde{\zeta}}$ of a vector multiplet on S^3, including a compensating gauge transformation to restore the Wess-Zumino gauge, are

$$\begin{aligned}
\delta A_\mu &= -i(\zeta\gamma^\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma^\mu\lambda) \\
\delta\sigma &= -\zeta\widetilde{\lambda} + \widetilde{\zeta}\lambda \\
\delta\lambda &= +\zeta\left(D + \frac{\sigma}{\ell}\right) - \frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho\zeta - i(D_\mu\sigma)\gamma^\mu\zeta \\
\delta\widetilde{\lambda} &= -\widetilde{\zeta}\left(D + \frac{\sigma}{\ell}\right) - \frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho\widetilde{\zeta} + i(D_\mu\sigma)\gamma^\mu\widetilde{\zeta} \\
\delta D &= iD_\mu(\zeta\gamma^\mu\widetilde{\lambda} - \widetilde{\zeta}\gamma^\mu\lambda) + \frac{1}{\ell}(\zeta\widetilde{\lambda} - \widetilde{\zeta}\lambda) - i(\widetilde{\zeta}[\sigma,\lambda] - \zeta[\sigma,\widetilde{\lambda}])
\end{aligned} \tag{5.5}$$

where $F_{\mu\nu}$ is the field strength of A_μ and $D_\mu = \nabla_\mu - iA_\mu$ is the general and gauge covariant derivative. The vector multiplet has two possible supersymmetric kinetic Lagrangians in three dimensions: the supersymmetric Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = \frac{1}{e^2}\mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\sigma D^\mu\sigma - 2i\widetilde{\lambda}\gamma^\mu D_\mu\lambda - 2i\widetilde{\lambda}[\sigma,\lambda] + \left(D + \frac{\sigma}{\ell}\right)^2 + \frac{\widetilde{\lambda}\lambda}{\ell}\right) \tag{5.6}$$

with gauge coupling e, and the supersymmetric Chern-Simons Lagrangian

$$\mathcal{L}_{CS} = \frac{ik}{4\pi}\mathrm{Tr}\left(\varepsilon^{\mu\nu\rho}A_\mu(\partial_\nu A_\rho + \frac{2i}{3}A_\nu A_\rho) + 2D\sigma + 2\widetilde{\lambda}\lambda\right) \tag{5.7}$$

with quantised level k. If the gauge group is $U(N)$ we can also write a Fayet-Iliopoulos Lagrangian

$$\mathscr{L}_{FI} = i\xi \,\mathrm{Tr}\left(D - \frac{\sigma}{\ell}\right) \tag{5.8}$$

with parameter ξ.

Matter fields are packaged in a *chiral multiplet*

$$\Phi = (\phi, \psi_\alpha, F) \tag{5.9}$$

containing a complex scalar ϕ, a fermion ψ and a complex auxiliary field F, and the conjugate *antichiral multiplet*

$$\widetilde{\Phi} = (\widetilde{\phi}, \widetilde{\psi}_\alpha, \widetilde{F}) \ . \tag{5.10}$$

The chiral multiplet can carry R-charge r, central charge z and transform in a representation of the gauge group.

Taking into account the compensating gauge transformation that restores Wess-Zumino gauge, the supersymmetry transformations of the charged chiral multiplet Φ are

$$\begin{aligned}
\delta\phi &= \sqrt{2}\zeta\psi \\
\delta\psi &= \sqrt{2}\zeta F - \sqrt{2}i\left(z - \sigma + i\frac{r}{\ell}\right)\widetilde{\zeta}\phi - \sqrt{2}i\gamma^\mu\widetilde{\zeta}D_\mu\phi \\
\delta F &= \sqrt{2}i\left(z - \sigma + i\frac{r-2}{\ell}\right)\widetilde{\zeta}\psi + 2i\widetilde{\zeta}\lambda\phi - \sqrt{2}iD_\mu(\widetilde{\zeta}\gamma^\mu\psi)
\end{aligned} \tag{5.11}$$

where the fields in the vector multiplet act on the fields in the chiral multiplet in the appropriate representation. The supersymmetry transformations for $\widetilde{\Phi}$ can be obtained by conjugation. I will again define the path integration contour for Φ and $\widetilde{\Phi}$ by imposing that they are complex conjugate as in Minkowskian signature. Note that, as in flat space, the central charge z, which gives a real mass to the chiral multiplet, behaves like the real scalar σ. In fact z can be viewed as the σ component of a background Abelian vector multiplet, and receives contributions from the global symmetries under which the chiral multiplet is charged.

The gauge invariant kinetic Lagrangian for the chiral multiplet Φ and its conjugate $\widetilde{\Phi}$ is

$$\begin{aligned}
\mathscr{L}_{mat} = {} & D_\mu\widetilde{\phi}D^\mu\phi - i\widetilde{\psi}\gamma^\mu D_\mu\psi + \widetilde{F}F - i\widetilde{\phi}\left(D + \frac{\sigma}{\ell}\right)\phi + 2i\frac{r-1}{\ell}\widetilde{\phi}(z-\sigma)\phi + \\
& + \widetilde{\phi}\left((z-\sigma)^2 - \frac{r(r-2)}{\ell^2}\right)\phi + i\widetilde{\psi}\left(z - \sigma + \frac{i}{\ell}\left(r - \frac{1}{2}\right)\right)\psi + \sqrt{2}i(\widetilde{\phi}\lambda\psi + \phi\widetilde{\lambda}\widetilde{\psi}).
\end{aligned} \tag{5.12}$$

Chiral multiplets can also interact via F-term Lagrangian derived from a superpotential $W(\Phi)$, a gauge invariant holomorphic function of the chiral superfields with R-charge 2 and central charge 0. Recall that in flat space $\mathscr{L}_W = \int d^2\theta\, W(\Phi) + \int d^2\widetilde{\theta}\, \widetilde{W}(\widetilde{\Phi}) = F_{W(\Phi)} + \widetilde{F}_{\widetilde{W}(\widetilde{\Phi})}$. Specialising the supersymmetry variation δF (5.11) to a gauge invariant chiral multiplet of R-charge 2 and zero central charge, we see that it is a total derivative, thus the Lagrangian

$$\mathscr{L}_W = F_{W(\Phi)} + \widetilde{F}_{\widetilde{W}(\widetilde{\Phi})} = \left(F\frac{\partial W}{\partial\phi} + \psi\psi\frac{\partial^2 W}{\partial\phi^2}\right) + \left(\widetilde{F}\frac{\partial\widetilde{W}}{\partial\widetilde{\phi}} + \widetilde{\psi}\widetilde{\psi}\frac{\partial^2\widetilde{W}}{\partial\widetilde{\phi}^2}\right) \tag{5.13}$$

is also supersymmetric in curved space.

5.2 Localisation of 3d $\mathcal{N}=2$ gauge theories on S^3

Now that we have all the ingredients of $\mathcal{N}=2$ supersymmetric gauge theories on S^3, we can apply the methods of section 3 to localise the partition function of these theories, which take the schematic form

$$Z[\hat{V}] = \int [\mathscr{D}\Phi][\mathscr{D}V]\; e^{-(S_{YM}[V]+S_{CS}[V]+S_{FI}[V]+S_{mat}[\Phi,V,\hat{V}]+S_W[\Phi])} \;. \tag{5.14}$$

Here Φ are chiral multiplets and V are dynamical vector multiplets for the gauge group G, over which we path integrate. $\hat{V}$ are background vector multiplets associated to the global symmetry group $\hat{G}$, which are not path integrated over.

To start, we must select the supercharge $\mathscr{Q}$ that will use to localise. We will use a supercharge Q associated to a Killing spinor ζ_α and a conjugate supercharge $\widetilde{Q}$ associated to $\widetilde{\zeta}_\alpha = \varepsilon_{\alpha\beta}(\zeta^\dagger)^\beta$, normalised as $\widetilde{\zeta}\zeta = |\zeta|^2 = 1$, such that the spinor bilinear Killing vector is the unit norm vector

$$K = \widetilde{\zeta}\gamma^\mu\zeta\;\partial_\mu = \frac{2}{\ell}\,\partial_\psi \;. \tag{5.15}$$

In the frame (5.1), we choose $\zeta = \begin{pmatrix}1\\0\end{pmatrix}$, $\widetilde{\zeta} = \begin{pmatrix}0\\1\end{pmatrix}$, so the spinor bilinear Killing vector is $(K^a) = (0,0,1)$. This supercharge was chosen in [19] because it preserves a half-BPS supersymmetric Wilson loop along the great circles which are integral curves of (5.15).

Then we look at the actions associated to the Lagrangians (5.6)-(5.8), (5.12)-(5.13). Being supersymmetric, they are closed under δ_ζ and $\delta_{\widetilde{\zeta}}$. It turns out that the kinetic actions S_{YM}, S_{mat} are also Q- and $\widetilde{Q}$-exact,

$$\begin{aligned} S_{YM} &= \delta_{\widetilde{\zeta}}\delta_\zeta \int \sqrt{g}\,(\widetilde{\lambda}\lambda - 2D\sigma) = \delta_\zeta\delta_{\widetilde{\zeta}} \int \sqrt{g}\,(\widetilde{\lambda}\lambda - 2D\sigma) \\ S_{mat} &= \delta_{\widetilde{\zeta}}\delta_\zeta \int \sqrt{g}\,\left(\widetilde{\psi}\psi - 2i\widetilde{\phi}\sigma\phi + 2\frac{r-1}{\ell}\widetilde{\phi}\phi\right) = \\ &= \delta_\zeta\delta_{\widetilde{\zeta}} \int \sqrt{g}\,\left(\widetilde{\psi}\psi - 2i\widetilde{\phi}\sigma\phi + 2\frac{r-1}{\ell}\widetilde{\phi}\phi\right) \;, \end{aligned} \tag{5.16}$$

so they do not contribute to the partition function (5.14) by the argument (3.4). The F-term of a superpotentials is also Q-exact

$$F_W = \delta_\zeta\left(\frac{\psi}{\sqrt{2}\zeta}\right) \tag{5.17}$$

due to (5.11) for $r=2$, $z=0$ and no coupling to the vector multiplet. Similarly the $\widetilde{F}$-term is $\widetilde{Q}$-exact. We conclude that the path integral (5.14) is independent of the Yang-Mills coupling e and of all superpotential couplings. On the other hand, the supersymmetric Chern-Simons action (5.7) and Fayet-Iliopoulos action (5.8) are neither Q or $\widetilde{Q}$-exact, so the path integral depends on the Chern-Simons level k and the FI parameter ξ.

Next, we define Q- and $\widetilde{Q}$-exact localising actions, that we will use to deform the path integral as in (3.8) without affecting the result, as long as we compute Q- or $\widetilde{Q}$-closed observables. In the language of section 3.2, we use a real localising supercharge $\mathscr{Q} = Q + \widetilde{Q}$. Both the kinetic actions

(5.16) are exact, so they can act as localising actions. Alternatively we can make the canonical choice (3.12) with $\mathscr{Q} = Q + \widetilde{Q}$, leading to S_{YM} for the vector multiplet and to

$$S^{\Phi}_{loc} = \delta_{\zeta} \delta_{\widetilde{\zeta}} \int (\widetilde{\psi}\psi - 2i\widetilde{\phi}\sigma\phi) \tag{5.18}$$

for the chiral multiplet. The kinetic Lagrangian (5.12), which preserves all the four supersymmetries on S^3, is $SO(4)$ invariant, whereas the Lagrangian associated to (5.18) breaks part of the isometries by choosing the localising Killing spinors, which select a preferred direction (5.15). (5.12) and (5.18) lead to the same localisation locus and 1-loop determinant, even though the 1-loop kinetic operators in (3.16) are different in the two cases.

Having chosen the localising actions (3.11), we look at their saddle points, which dominate the path integral in the limit where the localisation parameter $t \to \infty$ as in (3.10). These saddle points coincide with the BPS configurations (3.7) for the localising supercharge.

The saddle points of the Yang-Mills action for the vector multiplet are

$$\begin{aligned} \sigma &= const., \qquad & D &= -\frac{\sigma}{\ell} , \\ A_\mu &= 0 , \qquad & \lambda &= \widetilde{\lambda} = 0 . \end{aligned} \tag{5.19}$$

Note that (5.19) also apply to background vector multiplets, being just the supersymmetry conditions on the background. In the following I will use gauge transformations to diagonalise σ so that it belongs to the Cartan subalgebra of the Lie algebra of the gauge group. I will call its eigenvalues σ_i.

The saddle points of the localising action for the chiral multiplet, be it (5.12) or (5.18), consist of vanishing field configurations

$$\phi = \psi = F = 0 . \tag{5.20}$$

This result is less obvious, but one can show that any non-trivial saddles for the bosons in the chiral multiplet are singular, hence they must be discarded.

So the saddle points of the localising action involve vanishing chiral multiplets and are determined by a constant background for the real scalar σ in the vector multiplet. From this we can already conclude that the path integral of $3d$ $\mathscr{N} = 2$ supersymmetric gauge theories on S^3 reduces to an ordinary matrix integral on constant σ matrices.

Because in flat space σ parametrizes (along with dual photons) the Coulomb branch of the moduli space of supersymmetric vacua of $3d$ $\mathscr{N} = 2$ gauge theories, this standard choice of localisation scheme is sometimes called *localisation on the Coulomb branch*.

Having determined the saddle point configurations X_0 of the localising action, we now have to evaluate the classical actions on the saddles to get the factor $S[X_0]$ in (3.17). On the saddle point configurations $S_{YM} = S_{mat} = S_W = 0$, as expected since these actions are $\mathscr{Q}$-exact. For the Fayet-Iliopoulos and Chern-Simons actions, using $\mathrm{Vol}(S^3) = 2\pi^2\ell^3$, we obtain

$$\begin{aligned} S_{FI}[X_0] &= -i4\pi^2(\xi\ell)\,\mathrm{Tr}(\ell\sigma) = -i4\pi^2(\xi\ell)\sum_i(\ell\sigma_i) \\ S_{CS}[X_0] &= -ik\pi\,\mathrm{Tr}(\ell\sigma)^2 = -ik\pi(\xi\ell)\sum_i(\ell\sigma_i)^2 . \end{aligned} \tag{5.21}$$

Finally we need to compute the determinants of the exact "1-loop" operator $\frac{\delta^2 S_{loc}[X]}{\delta X^2}\big|_{X=X_0}$ for the fluctuations of the chiral and vector multiplet (including ghosts). This is the hardest part of the localisation computation, but it is conceptually simple: we just need to diagonalise an operator. The details of the computation can be found in [19, 30, 31]. Here I will quote the results and then comment on how supersymmetry can be exploited to simplify the computation of the 1-loop determinant, following [32].

The 1-loop determinant for a chiral multiplet is

$$Z^{\Phi}_{1-loop}(m+ir) = \prod_{n=1}^{\infty} \left(\frac{n+1+im-r}{n+1-im+r} \right)^n = \Gamma_h(m+ir) , \tag{5.22}$$

where m is the effective real mass of the chiral multiplet in units of the inverse radius $1/\ell$. If Φ transforms in a nontrivial representation $(\mathscr{R}_\Phi, \widetilde{\mathscr{R}}_\Phi)$ of the gauge and flavour group $G \times \hat{G}$ with weights ρ and $\hat{\rho}$, then for each component the effective real mass is

$$m \equiv \ell(\rho(\sigma) + \hat{\rho}(\hat{\sigma})) \tag{5.23}$$

in terms of the real scalars σ, $\hat{\sigma}$ in the dynamical vector multiplet associated to the gauge group G and in the background vector multiplet associated to the global symmetry group $\hat{G}$. The 1-loop determinant of the whole chiral multiplet is a product of 1-loop determinants (5.22) for each component of effective real mass (5.23). In these formulae I have absorbed the central charge z in $\hat{\rho}(\hat{\sigma})$. The infinite product in (5.22) is divergent and requires regularisation. It is usually defined in zeta function regularisation, yielding the hyperbolic gamma function Γ_h, with periods (i,i). I refer to Antonio Amariti's lectures in this school or to appendix A of [33] for more details on hyperbolic gamma functions. In physics terms this is an ultraviolet divergence. In (5.22) we are identifying the UV subtraction scale with the IR scale $1/\ell$ of the field theory on S^3.

The 1-loop determinant for a vector multiplet (including Faddeev-Popov ghosts) is

$$Z^{V}_{1-loop}(\sigma) = \prod_{\alpha\in\Delta_+} \frac{\sinh^2(\pi\ell\alpha(\sigma))}{(\pi\ell\alpha(\sigma))^2} = \prod_{\alpha\in\Delta_+} \frac{1}{(\pi\ell\alpha(\sigma))^2 \Gamma_h(\alpha(\sigma))\Gamma_h(-\alpha(\sigma))} , \tag{5.24}$$

where the product is over the set of positive roots Δ_+ of the Lie algebra of the gauge group G, corresponding to the superpartners of the W-bosons. The superpartners of the photons, which are elements of the Cartan subalgebra, are zero-modes and need to be integrated over.

As mentioned above, I have chosen a diagonal gauge where σ belongs to the Cartan subalgebra of the Lie algebra of G. The Jacobian of the change of variables is the usual Vandermonde determinant

$$|J| = \prod_{\alpha\in\Delta_+} (\pi\ell\alpha(\sigma))^2 \tag{5.25}$$

which exactly cancels a similar factor in (5.24).

5.2.1 Intermezzo on 1-loop determinants

Let us look in more detail at the 1-loop operators for fluctuations of the components of the chiral multiplet, following [32]. Using the localising action (5.18), the 1-loop operators for the bosonic and fermionic fluctuations are respectively

$$\Delta_\phi = -D_\mu D^\mu - 2i(r-1)K^\mu D_\mu + \frac{r(2r-3)}{2\ell^2} + \frac{6r}{\ell^2} \tag{5.26}$$

$$\Delta_\psi = -i\gamma^\mu D_\mu + im - \frac{1}{2\ell} + (r-1)\gamma^\mu K_\mu \ . \tag{5.27}$$

The 1-loop determinant is $Z^\Phi_{1-loop} = \frac{\det\Delta_\psi}{\det\Delta_\phi}$. The diagonalisation of these operators is tedious, and moreover one eventually finds lots of cancellations, because most fermionic and bosonic eqigenstates are paired by supersymmetry. This pairing is seen as follows. First of all, one can show that

$$\Delta_\psi \Psi = M\Psi \quad \Rightarrow \quad \Delta_\phi(\widetilde{\zeta}\Psi) = M(M-2im)\cdot\widetilde{\zeta}\Psi \tag{5.28}$$

meaning that we can build a bosonic eigenstate $\widetilde{\zeta}\Psi$ of eigenvalue $M(M-2iM)$ from a fermionic eigenstate Ψ of eigenvalue M. Similarly

$$\Delta_\phi \Phi = M(M-2im)\Phi \quad \Rightarrow \quad \Delta_\psi \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix} 2im & -1 \\ -M(M-2iM) & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \tag{5.29}$$

with

$$\Psi_1 \equiv \zeta\Phi\ , \qquad \Psi_2 \equiv i\gamma^\mu\zeta D_\mu\Phi + i\zeta\left(m + i\frac{r}{\ell}\right)\Phi\ , \tag{5.30}$$

meaning that from a bosonic eigenstate of eigenvalue $M(M-2im)$ we can construct fermionic eigenstates of eigenvalues M and $-(M-2im)$, by diagonalising the matrix in (5.29).

Modes which participate in this pairing can be disregarded in the computation of the 1-loop determinant, because the bosonic and fermionic contributions cancel out. Therefore we only need to find the unpaired modes. These come in two flavours:

1. Spinor eigenmodes Ψ such that $\widetilde{\zeta}\Psi = 0$, meaning $\Psi \propto \widetilde{\zeta}\widetilde{F}$, with $\widetilde{F}$ a scalar of R-charge $2-r$. These unpaired fermionic eigenmodes contribute to the numerator of (5.22).

2. Scalar eigenmodes Φ such that $\Psi_1 = \zeta\Phi \propto \Psi_2 = i\gamma^\mu\zeta D_\mu\Phi + i\zeta\left(m + i\frac{r}{\ell}\right)\Phi$. These unpaired bosonic eigenmodes contribute to the denominator of (5.22).

5.2.2 The partition function of $3d$ $\mathcal{N}=2$ gauge theories on S^3

We can finally put together the results of section 5.2 to obtain the localisation formula for the partition function (5.14) of a $3d$ $\mathcal{N}=2$ gauge theories on S^3. This is a function of the Fayet-Iliopoulos and Chern-Simons couplings ξ and k, of the R-charges r of the chiral multiplets, and of the constant real scalars $\hat{\sigma}$ in background vector multiplets, which account for bare real masses. In the following we will set the radius of the sphere ℓ to 1, or equivalently work with rescaled dimensionless quantities.

The localisation formula for the partition function is

$$Z(\hat{\sigma};k,\xi,r) = \frac{1}{|\mathscr{W}_G|}\int\left(\prod_{i=1}^{\mathrm{rk}(G)} d\sigma_i\right) Z_{class}(\sigma;\xi,k)\, Z_{1-loop}(\sigma,\hat{\sigma},r)\ , \tag{5.31}$$

where the integral is over real scalars in the Cartan subalgebra of the gauge group G, parametrising the localisation locus (5.19) in diagonal gauge, $|\mathscr{W}_G|$ is the order of the Weyl group, the residual gauge group in the diagonal gauge,

$$Z_{class}(\sigma;k,\xi) = e^{ik\pi\sum_i \sigma_i^2 + i4\pi^2\xi\sum_i \sigma_i} \tag{5.32}$$

is the contribution of the classical action evaluated on the localisation locus (5.19)-(5.20), depending on the Chern-Simons level k and the Fayet-Iliopoulos parameter ξ, and

$$Z_{1-loop}(\sigma,\hat{\sigma},r) = \prod_{\alpha\in\Delta_+} \sinh^2(\pi\ell\alpha(\sigma)) \cdot \prod_{\substack{\rho\in\mathscr{R}_\Phi \\ \hat{\rho}\in\hat{\mathscr{R}}_\Phi}} \Gamma_h(\rho(\sigma)+\hat{\rho}(\hat{\sigma})+ir) \tag{5.33}$$

is the contribution of 1-loop determinants for vector multiplets (including the Vandermonde determinant) and for chiral multiplets transforming in representations $(\mathscr{R}_\Phi, \hat{\mathscr{R}}_\Phi)$ of the gauge and flavour group $G\times\hat{G}$. These formulae are schematic: the gauge group could be a product of simple groups each with its own Chern-Simons and Fayet-Iliopoulos couplings, and similarly there could be various chiral multiplets transforming in different representations and with different R-charges. If so (5.31)-(5.33) become products of various factors, one for each ingredient.

The localisation formula (5.31)-(5.33) is the main and final result that I present in this course. Using supersymmetric localisation, we have managed to rewrite the exact path integral of a quantum $\mathscr{N}=2$ supersymmetric gauge theory on the 3-sphere as an ordinary integral for a matrix model. This dramatic simplification is due to the fact that the localisation locus (5.19)-(5.20) that contributes to the integral consists only of constant field configurations. Integrals of the type (5.31) can often be evaluated explicitly by closing the integration contour in the complex plane and using Cauchy's residue theorem, see for instance [34]. Otherwise one can use mathematical identities for the integrals, such as Cauchy's determinant identities or identities for hyperbolic hypergeometric integrals as those derived in [35], to prove the equality of the sphere partition functions of apparently different gauge theories [36, 37, 38, 33], thus providing non-trivial tests of non-perturbative infrared dualities [39, 40, 41, 42, 43, 33]. Note that even though we used the ultraviolet definition of the gauge theory to derive the localisation formula (5.31), the independence of the partition function on the Yang-Mills coupling, guaranteed by (5.16), ensures that the localised exact partition functions can be used to study the infrared fixed point of the gauge theory obtained as $e\to\infty$.

The localisation formula (5.31)-(5.33) for the partition function can be immediately generalised to include local or non-local $\mathscr{Q}$-closed operators, such as for instance the supersymmetric Wilson loops along great circles of S^3 that were studied in [19]. The insertion of such a Wilson loop in representation $\mathscr{R}$ in the path integral reduces, evaluating it on the localisation locus (5.19), to the insertion of

$$\mathrm{Tr}_{\mathscr{R}}\left(e^{2\pi\sigma}\right) \tag{5.34}$$

in the ordinary integral (5.31).

Another interesting generalisation is obtained by considering various squashed metrics on the 3-sphere, whose isometry groups are subgroups of $SO(4)$ [32, 44]. The only effect of these squashings is (at most) to change the periodicities of the hyperbolic double gamma functions Γ_h in (5.22) and the last expression in (5.24) from (i,i) to $(ib,i/b)$, where b is a squashing parameter that can be extracted from the background [45, 46]. This observation points towards the fact that rigid supersymmetric theories on curved manifolds appear to be quasi-topological, in the sense that they are insensitive to most deformations of the background and only depend on a few topological or geometric structures and invariants.

Before concluding, I should mention that the localisation methods that I have reviewed here for three-dimensional $\mathcal{N}=2$ theories on the round S^3 have been successfully applied to field theories with various amounts of supersymmetries on various manifolds of dimension from 2 to 5. Due to the abundant scientific production in this field, it would not be possible for me to provide a set of citations which is both concise and complete; instead, I invite interested readers to consult the list of citations to [4] for an overview.

This is an exciting time in our understanding of supersymmetric quantum field theories. Many new exact results are becoming available by applying localisation techniques to supersymmetric theories on curved manifolds, that I have reviewed in these lectures. I hope that this course will serve as a useful introduction to this rapidly developing field and will convince a few young researchers to work on this subject.

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